The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Canadian Senior Mathematics Contest

Thursday, November 20, 2014
(in North America and South America)

Friday, November 21, 2014
(outside of North America and South America)

Solutions

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Part A

1. We note that $\angle DAF = \angle FAB - \angle DAB$ and that $\angle FAB = \angle FAE + \angle EAB$.
Since $ABCD$ is a square, then $\angle DAB = 90^\circ$.
Since each of $\triangle AEF$ and $\triangle ABE$ is equilateral, then $\angle FAE = \angle EAB = 60^\circ$.
Therefore, $\angle DAF = (60^\circ + 60^\circ) - 90^\circ = 30^\circ$.

ANSWER: 30°

2. Since the ratio of the number of dimes to the number of quarters is $3 : 2$, then we let the numbers of dimes and quarters be $3k$ and $2k$, respectively, for some positive integer k.
Since each dime is worth 10 cents, then the $3k$ dimes are worth $10(3k) = 30k$ cents.
Since each quarter is worth 25 cents, then the $2k$ quarters are worth $25(2k) = 50k$ cents.
Since the total value of the coins is \$4 which equals 400 cents, then $30k + 50k = 400$ or $80k = 400$, and so $k = 5$.
Therefore, there are $3(5) = 15$ dimes in the jar.
(Note that 15 dimes are worth 150 cents and $2(5) = 10$ quarters are worth 250 cents, for a total of 400 cents, as required.)

ANSWER: 15

3. We note that $5000 = 5 \cdot 1000 = 5 \cdot 10^3 = 5 \cdot (2 \cdot 5)^3 = 5 \cdot 2^3 \cdot 5^3 = 2^3 5^4$.
Since neither m nor n is divisible by 10, then neither m nor n can have factors of both 2 and 5.
Since the product mn equals 5000, the only prime factors of 5000 are 2 and 5, and neither m nor n has factors of both 2 and 5, then m and n must be 2^3 and 5^4, or 5^4 and 2^3.
Therefore, $m + n = 2^3 + 5^4 = 8 + 625 = 633$.

ANSWER: 633

4. *Solution 1*
Since $f(x) + f(x + 3) = 2x + 5$ for all x, then using $x = 2$ we obtain $f(2) + f(5) = 2(2) + 5$ and so $f(2) + f(5) = 9$.
Also, using $x = 5$ we obtain $f(5) + f(8) = 2(5) + 5$, and so $f(5) + f(8) = 15$.
Adding these two equations, we obtain $f(2) + f(8) + 2f(5) = 24$.
Since $f(8) + f(2) = 12$, then $2f(5) = 24 - (f(2) + f(8)) = 24 - 12 = 12$ and so $f(5) = 6$.

Solution 2
The function $f(x) = x + 1$ satisfies the required properties, since

- $f(x + 3) = (x + 3) + 1 = x + 4$ and so $f(x) + f(x + 3) = (x + 1) + (x + 4) = 2x + 5$, and
- $f(2) = 2 + 1 = 3$ and $f(8) = 8 + 1 = 9$ gives $f(8) + f(2) = 12$.

So $f(x) = x + 1$ is a function that works and the question implies that the value of $f(5)$ is the same no matter which function f that satisfies the given properties is chosen.
Therefore, $f(5) = 5 + 1 = 6$.
(Note that there may be other functions that satisfy these relationships.)

ANSWER: 6

5. We use the following facts about exponents:

- $\sqrt[3]{a} = a^{1/3}$ for every real number a
- Every real number can be squared
- The cube root of every number exists
- $(b^c)^d = b^{cd} = b^{dc} = (b^d)^c$ whenever b^c and b^d and b^{cd} all exist
- $(mn)^p = m^p n^p$ whenever m^p and n^p and $(mn)^p$ all exist

Manipulating the given equation, we obtain

$$\begin{aligned}
\sqrt[3]{(2+x)^2} + 3\sqrt[3]{(2-x)^2} &= 4\sqrt[3]{4-x^2} \\
\left((2+x)^2\right)^{1/3} + 3\left((2-x)^2\right)^{1/3} &= 4\left(4-x^2\right)^{1/3} \\
(2+x)^{2/3} + 3(2-x)^{2/3} &= 4\left((2+x)(2-x)\right)^{1/3} \\
\left((2+x)^{1/3}\right)^2 + 3\left((2-x)^{1/3}\right)^2 &= 4(2+x)^{1/3}(2-x)^{1/3}
\end{aligned}$$

Letting $u = (2+x)^{1/3}$ and $v = (2-x)^{1/3}$, this equation becomes

$$\begin{aligned}
u^2 + 3v^2 &= 4uv \\
u^2 - 4uv + 3v^2 &= 0 \\
(u-v)(u-3v) &= 0
\end{aligned}$$

Therefore, $u = v$ or $u = 3v$.

If $u = v$, then $(2+x)^{1/3} = (2-x)^{1/3}$.
Cubing both sides gives $2 + x = 2 - x$ and so $2x = 0$ or $x = 0$.

If $u = 3v$, then $(2+x)^{1/3} = 3(2-x)^{1/3}$.
Cubing both sides gives $2+x = 3^3(2-x)$ and so $2+x = 54-27x$ or $28x = 52$ or $x = \frac{52}{28} = \frac{13}{7}$.

We verify these answers in the original equation.

If $x = 0$, the left side of the equation becomes

$$\sqrt[3]{(2+0)^2} + 3\sqrt[3]{(2-0)^2} = \sqrt[3]{4} + 3\sqrt[3]{4} = 4\sqrt[3]{4} = 4\sqrt[3]{4-0^2}$$

which equals the right side of the equation, and so $x = 0$ is a solution.

If $x = \frac{13}{7}$, the right side of the equation equals

$$4\sqrt[3]{4-\left(\tfrac{13}{7}\right)^2} = 4\sqrt[3]{4-\tfrac{169}{49}} = 4\sqrt[3]{\tfrac{196-169}{49}} = 4\sqrt[3]{\tfrac{27}{49}} = 4\tfrac{3}{\sqrt[3]{49}} = \tfrac{12}{\sqrt[3]{49}}$$

and the left side of the equation equals

$$\sqrt[3]{(2+\tfrac{13}{7})^2} + 3\sqrt[3]{(2-\tfrac{13}{7})^2} = \sqrt[3]{\left(\tfrac{27}{7}\right)^2} + 3\sqrt[3]{\left(\tfrac{1}{7}\right)^2} = \sqrt[3]{\tfrac{729}{49}} + 3\sqrt[3]{\tfrac{1}{49}} = \tfrac{9}{\sqrt[3]{49}} + 3\tfrac{1}{\sqrt[3]{49}} = \tfrac{12}{\sqrt[3]{49}}$$

(We used the fact that $729 = 9^3$.) Therefore, the left side equals the right side, as required, and so $x = \frac{13}{7}$ is also a solution.

Thus, the solutions to the original equation are $x = 0$ and $x = \frac{13}{7}$.

Answer: $0, \frac{13}{7}$

6. If m and n are even and odd or odd and even, then $m - n$ is odd and so the lockers numbered m and n must have different colours.
Thus, none of the even-numbered lockers (we call these "even lockers") can be painted the same colour as any of the odd-numbered lockers ("odd lockers").
(If m and n are both odd or both even, then $m - n$ is even, so there is no restriction.)
This means that we cannot use all three colours in painting the even lockers and we cannot use all three colours in painting the odd lockers, otherwise there would be no colours left to use for the other set.
Thus, one or two colours are used to paint the even lockers and one or two colours are used to paint the odd lockers.
Furthermore, we cannot use two colours to paint the even lockers and two colours to paint the odd lockers, otherwise there would be a colour that overlaps.
Thus, either one colour is used for the even lockers and one colour for the odd lockers (it is not required that all three colours be used), or one colour is used for the even lockers and two colours for the odd lockers, or two colours are used for the even lockers and one colour for the odd lockers.
We count the number of ways of painting in the first and second cases. The number of ways of painting in the third case will be equal to the number of ways from the second case (the roles of the even and odd lockers can be exchanged since there is an equal number of even lockers and odd lockers).

Case 1: One colour for even lockers and one colour for odd lockers
There are three choices of colour for the even lockers (any of the three colours).
There are then two choices of colour for the odd lockers (either of the remaining colours).
In this case, there are $3 \times 2 = 6$ ways of colouring the lockers.

Case 2: One colour for even lockers and two colours for odd lockers
There are three choices of colour for the even lockers.
For each of these three choices, the odd lockers must be painted with the two remaining colours (there is no choice of the colours), making sure to use each colour on at least one locker.
For each of the five odd lockers, there are two choices of colour, so there are 2^5 ways of painting the odd lockers using at most these two colours. We subtract 2 ways from this total: one for painting them all one colour and one for painting them all the other colour (since these was counted in Case 1). Thus, there are $32 - 2 = 30$ ways of painting the odd lockers using exactly two colours.
In total, there are thus $3 \times 30 = 90$ ways of painting the lockers in this case.

Case 3: Two colours for even lockers and one colour for odd lockers
As in case 2, there are 90 ways of painting the lockers.

In total, there are $6 + 90 + 90 = 186$ ways of painting the lockers.

Answer: 186

Part B

1. (a) From the given list, the numbers 5, 10, 15 form an arithmetic sequence with common difference 5. None of the other sets of three numbers form an arithmetic sequence.

(b) *Solution 1*

Moving from the 2nd term to the 4th term involves adding the common difference twice.
Since the difference between the 2nd term and the 4th term is $13 - 7 = 6$, then 2 times the common difference equals 6, which means that the common difference is 3.
Therefore, the 3rd term, q, is 3 more than 7, or $q = 10$, and the 1st term, p, is 3 less than 7, or $p = 4$.
Thus, $p = 4$ and $q = 10$.

Solution 2

Suppose that the given arithmetic sequence has common difference d.
In terms of p and d, the four terms are p, $p + d$, $p + 2d$, $p + 3d$.
Since the four terms are $p, 7, q, 13$, then $p + d = 7$ and $p + 3d = 13$.
Subtracting the first of these equations from the second, we obtain $2d = 6$ or $d = 3$.
Since $p + d = 7$ and $d = 3$, then $p = 4$.
Finally, $q = p + 2d = 4 + 2(3) = 10$.
Thus, $p = 4$ and $q = 10$.

(c) *Solution 1*

Moving from the 1st term to the 4th term involves adding the common difference 3 times.
Since the difference between the 1st term and the 4th term is $(a + 21) - a = 21$, then 3 times the common difference equals 21, which means that the common difference is 7.
The difference between the 3rd term, c, and the 1st term, a, is twice the common difference, or 14, and so $c - a = 14$.

Solution 2

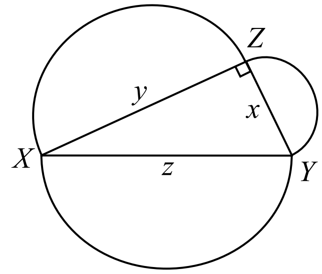
Suppose that the given arithmetic sequence has common difference d.
In terms of a and d, the four terms are a, $a + d$, $a + 2d$, $a + 3d$.
Since the four terms are $a, b, c, a + 21$, then $a + 3d = a + 21$ or $3d = 21$, and so $d = 7$.
Since $c = a + 2d$, then $c = a + 14$.
Finally, $c - a = (a + 14) - a = 14$.

(d) Since $(y-6), (2y+3), (y^2+2)$ form an arithmetic sequence in that order, then the differences between consecutive terms are equal and so

$$\begin{aligned}
(2y + 3) - (y - 6) &= (y^2 + 2) - (2y + 3) \\
y + 9 &= y^2 - 2y - 1 \\
0 &= y^2 - 3y - 10 \\
0 &= (y - 5)(y + 2)
\end{aligned}$$

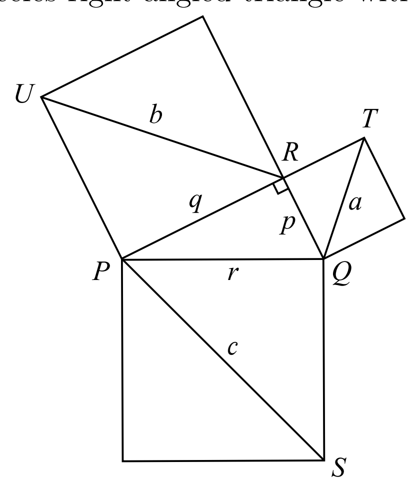
Therefore, $y = 5$ or $y = -2$.
We verify that each of these values of y gives an arithmetic sequence.
When $y = 5$, the sequence is $-1, 13, 27$ which is an arithmetic sequence with common difference 14.
When $y = -2$, the sequence is $-8, -1, 6$ which is an arithmetic sequence with common difference 7.

2. (a) (i) The semi-circles with diameter YZ, XZ and XY have diameters of length x, y and z, and so have radii of length $\frac{1}{2}x$, $\frac{1}{2}y$ and $\frac{1}{2}z$, respectively.



In terms of x, the area of a semi-circle with radius $\frac{1}{2}x$ is $\frac{1}{2}\pi(\frac{1}{2}x)^2 = \frac{1}{8}\pi x^2$. (This is one-half of the area of the circle with radius $\frac{1}{2}x$.)
Similarly, in terms of y and z, the areas of the semi-circles with diameter XZ and XY are $\frac{1}{8}\pi y^2$ and $\frac{1}{8}\pi z^2$, respectively.
Therefore, $\frac{1}{8}\pi x^2 = 50\pi$ and $\frac{1}{8}\pi y^2 = 288\pi$ and we want to determine $\frac{1}{8}\pi z^2$.
Since $\triangle XYZ$ is right-angled at Z, then $YZ^2 + XZ^2 = XY^2$, by the Pythagorean Theorem, and so $x^2 + y^2 = z^2$.
Multiplying both sides of this equation by $\frac{1}{8}\pi$, we obtain $\frac{1}{8}\pi x^2 + \frac{1}{8}\pi y^2 = \frac{1}{8}\pi z^2$.
Therefore, $50\pi + 288\pi = \frac{1}{8}\pi z^2$ and so $\frac{1}{8}\pi z^2 = 338\pi$.
Thus, the area of the semi-circle with diameter XY is $\frac{1}{8}\pi z^2 = 338\pi$.
(Note that we did not need to determine x, y or z.)

(ii) A square with side length s has diagonal of length $\sqrt{2}s$. This is because the diagonal is the hypotenuse of an isosceles right-angled triangle with equal sides of length s.



Therefore, $a = \sqrt{2}p$, $b = \sqrt{2}q$ and $c = \sqrt{2}r$.
Since $\triangle PQR$ is right-angled at R, then $QR^2 + PR^2 = PQ^2$, by the Pythagorean Theorem, and so $p^2 + q^2 = r^2$.
Multiplying both sides of this equation by 2, we obtain $2p^2 + 2q^2 = 2r^2$, and so $(\sqrt{2}p)^2 + (\sqrt{2}q)^2 = (\sqrt{2}r)^2$ or $a^2 + b^2 = c^2$.
Thus, the lengths a, b and c satisfy the Pythagorean equation, which tells us that the triangle formed with side lengths a, b and c is a right-angled triangle.

Note that the Pythagorean Theorem has two parts:

Suppose that $\triangle XYZ$ has side lengths $YZ = x$, $XZ = y$ and $XY = z$. Then:

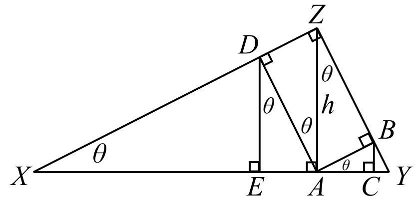
- If the triangle is right-angled at Z, then $x^2 + y^2 = z^2$, and
- If $x^2 + y^2 = z^2$, then the triangle is right-angled at Z.

(b) *Solution 1*

Let $AZ = h$ and let $\angle ZXY = \theta$.
Since $\triangle XDE$ is right-angled at E, then

$$\angle XDE = 180^\circ - \angle DXE - \angle XED = 180^\circ - \theta - 90^\circ = 90^\circ - \theta$$

Since $\angle XDA = 90^\circ$, then $\angle EDA = 90^\circ - \angle XDE = 90^\circ - (90^\circ - \theta) = \theta$.
Similarly, $\angle EAD = 90^\circ - \theta$, $\angle DAZ = \theta$, $\angle DZA = 90^\circ - \theta$, $\angle AZB = \theta$, $\angle ZAB = 90^\circ - \theta$, and $\angle BAC = \theta$.



Since $AZ = h$ and $\triangle ADZ$ is right-angled at D, then $AD = AZ\cos(\angle DAZ) = h\cos\theta$.
Since $AD = h\cos\theta$ and $\triangle DEA$ is right-angled at E, then

$$AE = AD\sin(\angle EDA) = (h\cos\theta)\sin\theta = h\cos\theta\sin\theta$$

Since $AZ = h$ and $\triangle ZBA$ is right-angled at B, then $AB = AZ\sin(\angle AZB) = h\sin\theta$.
Since $AB = h\sin\theta$ and $\triangle ACB$ is right-angled at C, then

$$AC = AB\cos(\angle BAC) = (h\sin\theta)\cos\theta = h\cos\theta\sin\theta$$

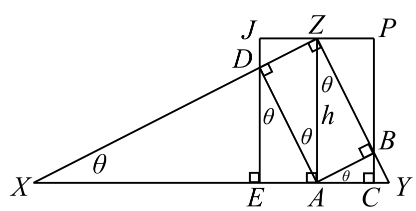
Therefore, $AE = AC$.

Solution 2

Let $AZ = h$ and let $\angle ZXY = \theta$.
Since $\triangle XDE$ is right-angled at E, then

$$\angle XDE = 180^\circ - \angle DXE - \angle XED = 180^\circ - \theta - 90^\circ = 90^\circ - \theta$$

Since $\angle XDA = 90^\circ$, then $\angle EDA = 90^\circ - \angle XDE = 90^\circ - (90^\circ - \theta) = \theta$.
Similarly, $\angle EAD = 90^\circ - \theta$, $\angle DAZ = \theta$, $\angle DZA = 90^\circ - \theta$, $\angle AZB = \theta$, $\angle ZAB = 90^\circ - \theta$, and $\angle BAC = \theta$.
Construct rectangle $JPCE$ with side JP passing through Z, J on ED extended and P on CB extended.



Now $ABZD$, $JZAE$ and $ZPCA$ are all rectangles since they each have three right angles.
Therefore, $JZ = EA$. We will show that $JZ = AC$, which will tell us that $AE = AC$.
Since $ABZD$ is a rectangle, then $AB = DZ$.
Since JZ is parallel to AC and DZ is parallel to AB, then $\angle JZD = \angle BAC$.
Since DJ is parallel to CB and DZ is parallel to AB, then $\angle JDZ = \angle CBA$.
Therefore, $\triangle ZDJ$ is congruent to $\triangle ABC$ (angle-side-angle).
Thus, $JZ = CA$ and so $AE = JZ = AC$, as required.

3. (a) Suppose that $x = \sqrt{a}$ is a solution of $x^2 = 3\lfloor x \rfloor + 1$ for some positive integer a.
Since $\sqrt{a} \geq 1$, then there exists a positive integer n for which $n \leq \sqrt{a} < n + 1$, which means that $\lfloor \sqrt{a} \rfloor = n$.
Substituting into the equation $x^2 = 3\lfloor x \rfloor + 1$, we obtain $(\sqrt{a})^2 = 3n + 1$ or $a = 3n + 1$.
Furthermore, squaring the inequality $n \leq \sqrt{a} < n + 1$, we obtain $n^2 \leq a < (n + 1)^2$ or $n^2 \leq a < n^2 + 2n + 1$. (Since $n > 0$, the inequalities are preserved.)
Since $a = 3n + 1$ and $n^2 \leq a < n^2 + 2n + 1$, then $n^2 \leq 3n + 1 < n^2 + 2n + 1$, which gives $n^2 - 3n - 1 \leq 0 < n^2 - n$.
This inequality is equivalent to the statement that $n^2 - 3n - 1 \leq 0$ and $0 < n^2 - n$.

The roots of the quadratic equation $n^2 - 3n - 1 = 0$ are $n = \dfrac{3 - \sqrt{13}}{2} \approx -0.303$ and $n = \dfrac{3 + \sqrt{13}}{2} \approx 3.303$.
To have $n^2 - 3n - 1 \leq 0$, we want n to be between these two roots (since the function $f(n) = n^2 - 3n - 1$ represents a parabola opening upwards); since n is an integer, then the interval narrows further to $0 \leq n \leq 3$.

The roots of the quadratic equation $n^2 - n = 0$ are $n = 0$ and $n = 1$.
To have $n^2 - n > 0$, we want n to be "outside" these two roots; therefore, $n < 0$ or $n > 1$.

Thus, we need $0 \leq n \leq 3$ and either $n < 0$ or $n > 1$.
Since n is an integer, then $n = 2$ or $n = 3$.
If $n = 2$, then $a = 3n + 1$ gives $a = 7$, which is the solution that we were given.
If $n = 3$, then $a = 3n + 1$ gives $a = 10$, which is the solution that we want.
Checking, we see that if $x = \sqrt{10}$, then $3 \leq x < 4$, so $3\lfloor x \rfloor + 1 = 3(3) + 1 = 10 = x^2$, as required.
Therefore, the value of a is 10.

(b) Suppose that x is a real number with $\lfloor x \rfloor = n$.
Then $n \leq x < n + 1$.
Squaring each part, we obtain $n^2 \leq x^2 < n^2 + 2n + 1$; since $n > 0$, the inequalities are preserved.
Therefore, $n^2 - 3\lfloor x \rfloor \leq x^2 - 3\lfloor x \rfloor < n^2 + 2n + 1 - 3\lfloor x \rfloor$.
Since $\lfloor x \rfloor = n$, then $n^2 - 3n \leq x^2 - 3\lfloor x \rfloor < n^2 + 2n + 1 - 3n$ or $n^2 - 3n \leq x^2 - 3\lfloor x \rfloor < n^2 - n + 1$.
Since $x^2 - 3\lfloor x \rfloor$ is an integer, then $n^2 - 3n \leq x^2 - 3\lfloor x \rfloor \leq n^2 - n$.
Therefore, $x^2 - 3\lfloor x \rfloor$ must lie between the integers $n^2 - 3n$ and $n^2 - n$, inclusive.
We can show that $x^2 - 3\lfloor x \rfloor$ can achieve each of these values by considering the values $x = \sqrt{m}$ for $m = n^2, n^2 + 1, \ldots, n^2 + 2n$.

(c) First, we note that for any integer k with $k \geq 0$, any real number x that is a solution to the equation $x^2 = 3\lfloor x \rfloor + k^2 - 1$ is of the form $x = \sqrt{a}$ or $x = -\sqrt{a}$ for some non-negative integer a. This is because if x is a solution, then the right side $3\lfloor x \rfloor + k^2 - 1$ is an integer, so x^2 is an integer which must be non-negative, say $x^2 = a$, which means that $x = \pm\sqrt{a}$.
We break our solution into four cases: $k = 0$, $k = 1$, $k = 2$, and $k \geq 3$.
Throughout this solution, a always represents a non-negative integer and n an integer.

<u>Case 1: $k = 0$</u>
We solve $x^2 = 3\lfloor x \rfloor - 1$.
First, we look for solutions $x = \sqrt{a}$ with $a \geq 0$.
Since $x \geq 0$, then $n \leq \sqrt{a} < n + 1$ for some non-negative integer n; thus, $\lfloor x \rfloor = \lfloor \sqrt{a} \rfloor = n$.
Since $x = \sqrt{a}$ is a solution, then $(\sqrt{a})^2 = 3n - 1$, or $a = 3n - 1$.

Since $n \le \sqrt{a} < n+1$ and $n > 0$, then $n^2 \le a < n^2 + 2n + 1$.
Therefore, $n^2 \le 3n - 1 < n^2 + 2n + 1$ and so $n^2 - 3n + 1 \le 0$ and $n^2 - n + 2 > 0$.
The roots of $n^2 - 3n + 1 = 0$ are $n = \dfrac{3-\sqrt{5}}{2} \approx 0.382$ and $n = \dfrac{3+\sqrt{5}}{2} \approx 2.618$.
The solution to $n^2 - 3n + 1 \le 0$ is $\dfrac{3-\sqrt{5}}{2} \le n \le \dfrac{3+\sqrt{5}}{2}$ (n is between the roots).
Since n is an integer with $n^2 - 3n + 1 \le 0$, then $1 \le n \le 2$.
The equation $n^2 - n + 2 = 0$ has no real roots and the corresponding parabola lies entirely above the horizontal axis, so every integer n satisfies $n^2 - n + 2 > 0$.
Therefore, the integers n that satisfy the inequalities $n^2 - 3n + 1 \le 0$ and $n^2 - n + 2 > 0$ are $n = 1$ and $n = 2$.
When $n = 1$, we have $a = 3(1) - 1 = 2$, and so $x = \sqrt{2}$ is a solution to the equation.
When $n = 2$, we have $a = 3(2) - 1 = 5$, and so $x = \sqrt{5}$ is a solution to the equation.
Next, we look for solutions $x = -\sqrt{a}$ with $a > 0$.
Since $x < 0$, then $n \le -\sqrt{a} < n + 1$ for some negative integer n.
Thus, $\lfloor x \rfloor = \lfloor -\sqrt{a} \rfloor = n$.
Since $x = -\sqrt{a}$ is a solution, then $(-\sqrt{a})^2 = 3n - 1$, or $a = 3n - 1$.
Since $n \le -\sqrt{a} < n + 1$ and $n < 0$, then $n^2 \ge a > n^2 + 2n + 1$.
Therefore, $n^2 \ge 3n - 1 > n^2 + 2n + 1$ and so $n^2 - 3n + 1 \ge 0$ and $n^2 - n + 2 < 0$.
The equation $n^2 - n + 2 = 0$ has no real roots and the corresponding parabola lies entirely above the horizontal axis, so no integer n satisfies $n^2 - n + 2 < 0$.
Therefore, there are no integers n that satisfy the inequalities $n^2 - 3n + 1 \ge 0$ and $n^2 - n + 2 < 0$, and so there are no solutions $x = -\sqrt{a}$.
Therefore, when $k = 0$, the solutions are $x = \sqrt{2}$ and $x = \sqrt{5}$.

<u>Case 2: $k = 1$</u>
We solve $x^2 = 3\lfloor x \rfloor$.
First, we look for solutions $x = \sqrt{a}$ with $\lfloor x \rfloor = \lfloor \sqrt{a} \rfloor = n$ for some non-negative integer n.
Proceeding as above, we obtain $a = 3n$ and the inequalities $n^2 - 3n \le 0$ and $n^2 - n + 1 > 0$.
The first inequality has integer solutions $n = 0, 1, 2, 3$ and the second inequality is true for all integers n.
Using $n = 0, 1, 2, 3$, we obtain the solutions $x = \sqrt{0}, \sqrt{3}, \sqrt{6}, \sqrt{9}$, or $x = 0, \sqrt{3}, \sqrt{6}, 3$.
Next, we look for solutions $x = -\sqrt{a}$ with $a > 0$ and $\lfloor x \rfloor = \lfloor -\sqrt{a} \rfloor = n$.
Proceeding as above, we obtain $a = 3n$ and the inequalities $n^2 - 3n \ge 0$ and $n^2 - n + 1 < 0$.
The inequality $n^2 - n + 1 < 0$ is not true for any integer n, so there are no solutions in this case.
Therefore, when $k = 1$, the solutions are $x = 0, \sqrt{3}, \sqrt{6}, 3$.

<u>Case 3: $k = 2$</u>
We solve $x^2 = 3\lfloor x \rfloor + 3$.
First, we look for solutions $x = \sqrt{a}$ with $\lfloor x \rfloor = \lfloor \sqrt{a} \rfloor = n$ for some non-negative integer n.
Proceeding as above, we obtain $a = 3n + 3$ and the inequalities $n^2 - 3n - 3 \le 0$ and $n^2 - n - 2 > 0$.
The first inequality has integer solutions $n = 0, 1, 2, 3$ and the second inequality is true when $n < -1$ or $n > 2$.
Therefore, these two inequalities are both true only when $n = 3$.
When $n = 3$, we obtain $a = 3(3) + 3 = 12$, and so $x = \sqrt{12}$ is a solution to the equation.
Next, we look for solutions $x = -\sqrt{a}$ with $a > 0$ and $\lfloor x \rfloor = \lfloor -\sqrt{a} \rfloor = n$.
Proceeding as above, we obtain $a = 3n + 3$ and the inequalities $n^2 - 3n - 3 \ge 0$ and $n^2 - n - 2 < 0$.

The second inequality is not true for any negative integer n (since its real number solution is $-1 < n < 2$), so there are no solutions in this case.
Therefore, when $k = 2$, the solution is $x = \sqrt{12}$.

Case 4: $k \geq 3$
We solve $x^2 = 3\lfloor x \rfloor + (k^2 - 1)$.
First, we look for solutions $x = \sqrt{a}$ with $\lfloor x \rfloor = \lfloor \sqrt{a} \rfloor = n$ for some non-negative integer n.
Proceeding as above, we obtain $a = 3n+(k^2-1)$ and the inequalities $n^2-3n-(k^2-1) \leq 0$ and $n^2 - n - (k^2 - 2) > 0$.
Consider the functions

$$f(n) = n^2 - 3n - (k^2 - 1) \qquad g(n) = n^2 - n - (k^2 - 2)$$

Note that $k^2 - 1 > 0$ and $k^2 - 2 > 0$ since $k \geq 3$.
Since the leading coefficient is positive and the constant term is negative in each case, then each of $f(n)$ and $g(n)$ has a positive zero and a negative zero.
Therefore, the integer solution set to the inequality $n^2 - 3n - (k^2 - 1) \leq 0$ will be of the form $C \leq n \leq D$ for some integers $C \leq 0 \leq D$. (Note that the real number solution set to the inequality $n^2 - 3n - (k^2 - 1) \leq 0$ is of the form $c \leq n \leq d$ for some real numbers $c < 0 < d$. When we restrict this set to integers, the solution set is of the form $C \leq n \leq D$ with $C \leq 0 \leq D$, where C is the smallest integer larger than c and D is the largest integer less than d.)
Also, the integer solution set to the inequality $n^2 - n - (k^2 - 2) > 0$ will be of the form $n \leq E$ or $n \geq F$ for some integers $E < 0 < F$. (Note that the real number solution set to the inequality $n^2 - n - (k^2 - 2) > 0$ is of the form $n < e$ or $n > f$ for some real numbers $e < 0 < f$. When we restrict this set to integers, the solution set is of the form $n \leq E$ or $n \geq F$, where E is the largest integer less than e and F is the smallest integer greater than f.)
Since we are currently restricting to the case that $n \geq 0$, then the set of integers n that satisfy these inequalities is of the form $0 \leq n \leq D$ and $n \geq F$.
We show that $n = k+1$ satisfies both inequalities, that $n = k+2$ does not satisfy the first inequality (thus $D = k + 1$), and that $n = k$ does not satisfy the second inequality (thus $F = k + 1$). Combining this information will tell us that $n = k + 1$ is the unique positive integer that satisfies these restrictions.
Now

$$\begin{aligned}
f(k+1) &= (k+1)^2 - 3(k+1) - (k^2 - 1) = k^2 + 2k + 1 - 3k - 3 - k^2 + 1 = -k - 1 < 0 \\
f(k+2) &= (k+2)^2 - 3(k+2) - (k^2 - 1) = k^2 + 4k + 4 - 3k - 6 - k^2 + 1 = k - 1 > 0 \\
g(k+1) &= (k+1)^2 - (k+1) - (k^2 - 2) = k^2 + 2k + 1 - k - 1 - k^2 + 2 = k + 2 > 0 \\
g(k) &= k^2 - k - (k^2 - 2) = -k + 2 < 0
\end{aligned}$$

(The final inequalities in each case come from the fact that $k \geq 3$.)
Therefore, $n = k + 1$ is the unique non-negative integer that satisfies the inequalities.
Since $n = k + 1$, then $a = 3n + (k^2 - 1) = 3(k + 1) + (k^2 - 1) = k^2 + 3k + 2$ and so $x = \sqrt{a} = \sqrt{k^2 + 3k + 2}$.

Next, we look for solutions $x = -\sqrt{a}$ with $a > 0$ and $\lfloor x \rfloor = \lfloor -\sqrt{a} \rfloor = n$ with $n < 0$.
Proceeding as above, we obtain $a = 3n+(k^2-1)$ and the inequalities $n^2-3n-(k^2-1) \geq 0$ and $n^2 - n - (k^2 - 2) < 0$.
Modelling the section of this case when $n \geq 0$, we find that the solution to the first inequality is of the form $n \leq G$ or $n \geq H$ for some integers $G < 0 < H$ and the solution

to the second inequality is of the form $J \leq n \leq L$ for some integers $J \leq 0 \leq L$.
Since we are considering the case that $n < 0$, then the set of integers n that satisfy these inequalities is of the form $n \leq G$ and $J \leq n < 0$.
We show that $n = -k+1$ satisfies both inequalities, that $n = -k+2$ does not satisfy the first inequality (thus $G = -k+1$), and that $n = -k$ does not satisfy the second inequality (thus $J = -k+1$). Combining this information will tell us that $n = -k+1$ is the unique possible negative value.
Now

$$\begin{aligned} f(-k+1) &= (-k+1)^2 - 3(-k+1) - (k^2-1) = k-1 > 0 \\ f(-k+2) &= (-k+2)^2 - 3(-k+2) - (k^2-1) = -k-1 < 0 \\ g(-k+1) &= (-k+1)^2 - (-k+1) - (k^2-2) = -k+2 < 0 \\ g(k) &= (-k)^2 - (-k) - (k^2-2) = k+2 > 0 \end{aligned}$$

(The final inequalities in each case come from the fact that $k \geq 3$.)
Therefore, $n = -k+1$ is the unique negative integer that satisfies the inequalities.
Since $n = -k+1$, then $a = 3n + (k^2-1) = 3(-k+1) + (k^2-1) = k^2-3k+2$ and so $x = -\sqrt{a} = -\sqrt{k^2-3k+2}$.
Therefore, in the case that $k \geq 3$, the solutions are $x = \sqrt{k^2+3k+2}$ and $x = -\sqrt{k^2-3k+2}$.

In summary, the solutions are given in the following table:

	x
$k=0$	$\sqrt{2}, \sqrt{5}$
$k=1$	$0, \sqrt{3}, \sqrt{6}, 3$
$k=2$	$\sqrt{12}$
$k \geq 3$	$\sqrt{k^2+3k+2}, \quad -\sqrt{k^2-3k+2}$