The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Thursday, November 20, 2014
(in North America and South America)

Friday, November 21, 2014
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

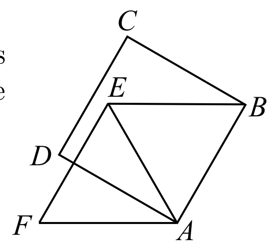
Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as $12.566\ldots$ or $4.646\ldots$.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. In the diagram, $ABCD$ is a square, $\triangle ABE$ is equilateral, and $\triangle AEF$ is equilateral. What is the measure of $\angle DAF$?



2. In a jar, the ratio of the number of dimes to the number of quarters is $3:2$. If the total value of these coins is \$4, how many dimes are in the jar?

   (Each dime is worth 10 cents, each quarter is worth 25 cents, and \$1 equals 100 cents.)

3. Positive integers m and n satisfy $mn = 5000$. If m is *not* divisible by 10 and n is *not* divisible by 10, what is the value of $m + n$?

4. A function f satisfies $f(x) + f(x+3) = 2x + 5$ for all x.
   If $f(8) + f(2) = 12$, determine the value of $f(5)$.

5. Determine all real numbers x for which $\sqrt[3]{(2+x)^2} + 3\sqrt[3]{(2-x)^2} = 4\sqrt[3]{4-x^2}$.

6. Ten lockers are in a row. The lockers are numbered in order with the positive integers 1 to 10. Each locker is to be painted either blue, red or green subject to the following rules:

   - Two lockers numbered m and n are painted different colours whenever $m - n$ is odd.
   - It is not required that all 3 colours be used.

   In how many ways can the collection of lockers be painted?

PART B

For each question in Part B, your solution must be well organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $5, 7, 9$ is an arithmetic sequence with three terms.

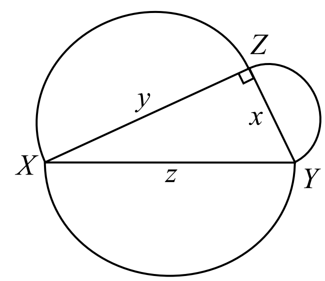
   (a) Three of the five numbers $2, 5, 10, 13, 15$ can be chosen to form an arithmetic sequence with three terms. What are the three numbers?

   (b) The numbers $p, 7, q, 13$, in that order, form an arithmetic sequence with four terms. Determine the value of p and the value of q.

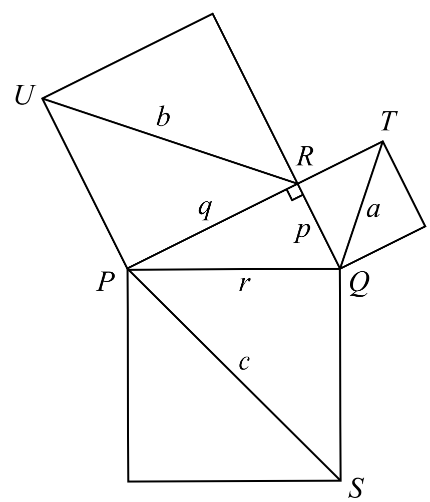
   (c) The numbers $a, b, c, (a + 21)$, in that order, form an arithmetic sequence with four terms. Determine the value of $c - a$.

   (d) The numbers $(y-6), (2y+3), (y^2+2)$, in that order, form an arithmetic sequence with three terms. Determine all possible values of y.

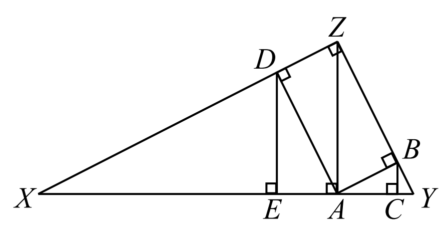
2. (a) (i) In the diagram, semi-circles are drawn on the sides of right-angled $\triangle XYZ$, as shown. If the area of the semi-circle with diameter YZ is 50π and the area of the semi-circle with diameter XZ is 288π, determine the area of the semi-circle with diameter XY.



   (ii) In the diagram, squares with side lengths p, q and r are constructed on the sides of right-angled $\triangle PQR$, as shown. Diagonals QT, RU and PS have lengths a, b and c, respectively. Prove that the triangle formed with sides of lengths a, b and c is a right-angled triangle.



   (b) In the diagram, $\triangle XYZ$ is right-angled at Z and AZ is the altitude from Z to XY. Also, segments AD and AB are altitudes in $\triangle AXZ$ and $\triangle AYZ$, respectively, and segments DE and BC are altitudes in $\triangle ADX$ and $\triangle ABY$, respectively. Prove that $AE = AC$.



3. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 4.2 \rfloor = 4$ and $\lfloor -2.4 \rfloor = -3$. That is, $\lfloor x \rfloor$ is the integer that satisfies the inequality $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

   (a) The equation $x^2 = 3\lfloor x \rfloor + 1$ has two solutions. One solution is $x = \sqrt{7}$. The second solution is of the form $x = \sqrt{a}$ for some positive integer a. Determine the value of a.

   (b) For each positive integer n, determine all possible integer values of the expression $x^2 - 3\lfloor x \rfloor$, where x is a real number with $\lfloor x \rfloor = n$.

   (c) For each integer k with $k \geq 0$, determine all real numbers x for which $x^2 = 3\lfloor x \rfloor + (k^2 - 1)$.