The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Galois Contest

Thursday, April 18, 2013
(in North America and South America)

Friday, April 19, 2013
(outside of North America and South America)

Solutions

1. (a) The slope of the line passing through the points $(2,0)$ and $(0,4)$ is $\dfrac{4-0}{0-2} = \dfrac{4}{-2} = -2$.

   Since the line passes through the point $(0,4)$, the y-intercept of this line is 4.
   Therefore, an equation of the line is $y = -2x + 4$.

   (b) Rearranging the equation from part (a), $y = -2x + 4$ becomes $2x + y = 4$.

   Dividing both sides of the equation by 4 we get $\dfrac{2x+y}{4} = \dfrac{4}{4}$ or $\dfrac{2x}{4} + \dfrac{y}{4} = 1$ and so the

   required form of the equation is $\dfrac{x}{2} + \dfrac{y}{4} = 1$.

   (c) To determine the x-intercept, we set $y = 0$ and solve for x.

   Thus, $\dfrac{x}{3} + \dfrac{y}{10} = 1$ becomes $\dfrac{x}{3} + \dfrac{0}{10} = 1$ or $\dfrac{x}{3} = 1$, and so $x = 3$.

   The x-intercept is 3.

   To determine the y-intercept, we let $x = 0$ and solve for y.

   Thus, $\dfrac{x}{3} + \dfrac{y}{10} = 1$, becomes $\dfrac{0}{3} + \dfrac{y}{10} = 1$ or $\dfrac{y}{10} = 1$, and so $y = 10$.

   The y-intercept is 10.
   (Note that the intercepts are the denominators of the two fractions.)

   (d) *Solution 1*

   The slope of the line passing through the points $(8,0)$ and $(2,3)$ is $\dfrac{3-0}{2-8} = \dfrac{3}{-6} = -\dfrac{1}{2}$.

   Thus, an equation of the line is $y = -\dfrac{1}{2}x + b$.

   To find the y-intercept b, we substitute $(8,0)$ into the equation and solve for b.

   The equation becomes, $0 = -\dfrac{1}{2}(8) + b$, or $0 = -4 + b$ and so $b = 4$.

   Therefore an equation of the line is $y = -\dfrac{1}{2}x + 4$.

   Rearranging this equation, $y = -\dfrac{1}{2}x + 4$ becomes $\dfrac{1}{2}x + y = 4$.

   Multiplying both sides of the equation by 2, we get $x + 2y = 8$.

   Dividing both sides of the equation by 8 we get, $\dfrac{x+2y}{8} = \dfrac{8}{8}$ or $\dfrac{x}{8} + \dfrac{2y}{8} = 1$ and so the

   required form of the equation is $\dfrac{x}{8} + \dfrac{y}{4} = 1$.

   Solution 2
   We recognize from the previous parts of the question that a line with equation written

   in the form $\dfrac{x}{e} + \dfrac{y}{f} = 1$, has x-intercept e and y-intercept f.

   Since the line passes through $(8,0)$, then its x-intercept is 8 and so $e = 8$.

   Substituting the point $(2,3)$ into the equation $\dfrac{x}{8} + \dfrac{y}{f} = 1$ gives $\dfrac{2}{8} + \dfrac{3}{f} = 1$ or $\dfrac{3}{f} = 1 - \dfrac{1}{4}$

   or $\dfrac{3}{f} = \dfrac{3}{4}$, and so $f = 4$.

   Therefore, the equation of the line is $\dfrac{x}{8} + \dfrac{y}{4} = 1$.

2. (a) *Solution 1*
A 100 cm tall red candle takes 600 minutes to burn completely.
Therefore, the red candle burns at a rate of $\dfrac{100 \text{ cm}}{600 \text{ min}} = \dfrac{1}{6}$ cm/min.
After 180 minutes, the height of the red candle will have decreased by $\dfrac{1}{6}$ cm/min $\times$ 180 min = 30 cm.

Solution 2
A 100 cm tall red candle takes 600 minutes to burn completely.
The fraction of the red candle that burns in 180 minutes is $\dfrac{180}{600} = \dfrac{3}{10}$.
Since the candle was initially 100 cm tall, then $\dfrac{3}{10} \times 100$ cm = 30 cm will have burned after 180 minutes.
Therefore, the height of the red candle will have decreased by 30 cm, 180 minutes after being lit.

(b) To reach a height of 80 cm, the green candle will have decreased by $100 - 80 = 20$ cm.
Since 20 cm is $\dfrac{20}{100} = \dfrac{1}{5}$ of the candle's original height, then it will take $\dfrac{1}{5}$ of the total time to decrease in height to this point.
Since it takes the green candle 480 minutes to burn completely, it will take $\dfrac{1}{5} \times 480 = 96$ minutes after being lit to decrease to a height of 80 cm.

(c) Since the red candle takes 600 minutes to burn completely, 60 minutes is $\dfrac{60}{600}$ or $\dfrac{1}{10}$ of its total burning time.
Therefore after 60 minutes, the red candle will have decreased by $\dfrac{1}{10} \times 100 = 10$ cm in height. That is, the red candle will be 90 cm tall after burning for 60 minutes.
Since the green candle takes 480 minutes to burn completely, 60 minutes is $\dfrac{60}{480}$ or $\dfrac{1}{8}$ of its total burning time.
Therefore after 60 minutes, the green candle will have decreased by $\dfrac{1}{8} \times 100 = 12.5$ cm in height. That is, the green candle will be 87.5 cm tall after burning for 60 minutes.
The red candle will be $90 - 87.5 = 2.5$ cm taller than the green candle 60 minutes after they are lit.

(d) *Solution 1*
From part (c), the green candle will decrease in height by 2.5 cm more than the red candle every 60 minutes (since their heights decrease at constant rates).
A difference of 2.5 cm in height every 60 minutes written as a fraction is $\dfrac{2.5}{60}$ cm/min., which is equivalent to $\dfrac{5}{120} = \dfrac{1}{24}$ cm/min.
That is, the green candle will decrease in height by 1 cm more than the red candle every 24 minutes after being lit.
Therefore, the red candle will be 7 cm taller than the green candle $7 \times 24 = 168$ minutes after they are lit.

Solution 2
The red candle burns at a rate of 100 cm every 600 minutes or $\frac{1}{6}$ cm/min.

The green candle burns at a rate of 100 cm every 480 minutes or $\frac{5}{24}$ cm/min.

In t minutes after being lit, $\frac{1}{6}t$ cm of the red candle will have burned.

In t minutes after being lit, $\frac{5}{24}t$ cm of the green candle will have burned.

Since both candles began with the same 100 cm height, then the red candle is 7 cm taller than the green candle when $(100 - \frac{1}{6}t) - (100 - \frac{5}{24}t) = 7$.

Simplifying this equation, we get $\frac{5}{24}t - \frac{1}{6}t = 7$ and by multiplying both sides by 24, $5t - 4t = 7 \times 24$, and so $t = 168$.

Therefore, the red candle is 7cm taller than the green candle 168 minutes after being lit.

3. (a) *Solution 1*

The last number in the 7th row is $7 \times 8 = 56$.

Since the 7th row has 7 numbers in it, we list the 7 even integers decreasing from 56, which are $56, 54, 52, 50, 48, 46, 44$.

Written in the order they will appear in the table, the numbers in the 7th row are, $44, 46, 48, 50, 52, 54, 56$.

Solution 2

The last number in the 6th row is $6 \times 7 = 42$.

Therefore the next even integer, 44, will appear as the first number in the 7th row of the table.

Since the 7th row has 7 numbers in it, we list the 7 even integers increasing from 44.

Thus the numbers in the 7th row are, $44, 46, 48, 50, 52, 54, 56$.

(b) The last number in the 100th row is $100 \times 101 = 10\,100$.

The last number in the 99th row is $99 \times 100 = 9900$.

Therefore the next even integer, 9902, will appear as the first number in the 100th row of the table.

The first and last numbers in the 100th row of the table are 9902 and $10\,100$, respectively.

(c) The last number in row r is equal to $r(r + 1)$, so $L = r(r + 1)$.

The first number in row $(r + 2)$ is 2 more than the last number in row $(r + 1)$.

The last number in row $(r + 1)$ is $(r + 1)(r + 2)$, so $F = (r + 1)(r + 2) + 2$.

We require $F + L$ to be at least 2013, so $F + L = (r + 1)(r + 2) + 2 + r(r + 1) \geq 2013$.

To determine the smallest value for r such that $F + L = (r+1)(r+2)+2+r(r+1) \geq 2013$, we solve the following inequality:

$$
\begin{aligned}
(r + 1)(r + 2) + 2 + r(r + 1) &\geq 2013 \\
r^2 + 3r + 2 + 2 + r^2 + r &\geq 2013 \\
2r^2 + 4r + 4 &\geq 2013 \\
r^2 + 2r + 2 &\geq 1006.5 \\
r^2 + 2r + 1 &\geq 1006.5 - 1 \\
(r + 1)^2 &\geq 1005.5 \\
\therefore \ r + 1 \geq +\sqrt{1005.5} \quad \text{or} \quad r + 1 &\leq -\sqrt{1005.5}
\end{aligned}
$$

Since r is positive, $r + 1 \geq \sqrt{1005.5}$ and so $r \geq +\sqrt{1005.5} - 1 \approx 30.7096$.

Thus, the smallest possible value of the integer r such that $F + L$ is at least 2013 is 31.

Check: Since L is the last number in row $r = 31$, then $L = 31 \times 32 = 992$.

Since F is the first number in row $r + 2 = 33$, then F is 2 more than the last number in row 32, or $F = (32 \times 33) + 2 = 1058$.

Therefore, $F + L = 1058 + 992 = 2050 \geq 2013$ as required.

We must also check if 31 is the smallest value of r such that $F + L \geq 2013$.

Since the numbers are arranged in the rows in a strictly increasing way, we need only check that when $r = 30$, $F + L < 2013$.

When $r = 30$, $F + L = ((31 \times 32) + 2) + (30 \times 31) = 994 + 930 = 1924 < 2013$.

4. (a) The shape formed by the water is a rectangular prism.

The area of the base of this rectangular prism is the same as the area of the base of the cube, which is $9 \times 9 = 81$ cm^2.

Since the height of the water is 1 cm, then the volume of water in the cube is $81 \times 1 = 81$ cm^3.

(b) When the cube is rotated $45°$ about edge PQ, edge MN will lie directly above PQ.

In this position, the water now has the shape of a triangular prism, as shown.

Moreover, the triangular face is a right-angled isosceles triangle. It is right-angled since the adjacent faces of the cube are perpendicular to one another, and it is isosceles as a result of the symmetry created by rotating the cube such that MN lies directly above PQ. (Put another way, by symmetry, the water level is such that $AP = CP$.)
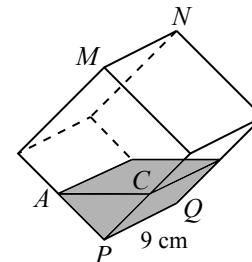
As shown in the second diagram, the depth of the water, h, is equal to the length of altitude PD in triangle APC.

In $\triangle APD$, $\angle DAP = 45°$ (since $\triangle APC$ is a right-angled isosceles triangle) and so $\angle APD = 180° - 90° - 45° = 45°$.

That is, $\triangle APD$ is also a right-angled isosceles triangle with $AD = DP = h$.
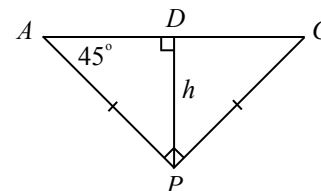
The volume of the water is given by the area of this right-angled isosceles triangle APC multiplied by the length of the prism, PQ, which is 9 cm.

Since $AD = DP = h$ and $\triangle ADP$ is congruent to $\triangle CDP$, then $AC = 2AD = 2h$.

Thus, the area of $\triangle APC$ is $\frac{1}{2}AC \times DP = \frac{1}{2}(2h) \times h = h^2$.

Therefore, the volume of water is equal to $(h^2 \times 9)$ cm^3.

In part (a), we found the volume of water to be equal to 81 cm^3.

Since no water has been lost, the volume is the same in this new orientation of the cube.

That is, $h^2 \times 9 = 81$ so $h^2 = 9$ and $h = 3$ (since $h > 0$).

Therefore, the depth of water in the cube is 3 cm.

(c) In this new position, the water now has the shape of a tetrahedron. By symmetry the water is the same distance "up" the edges and so $PR = PS = PT$.

Three of the faces of this tetrahedron, $\triangle PRS, \triangle PST, \triangle PTR$, are right triangles since adjacent edges of a cube are perpendicular ($\angle RPS = \angle SPT = \angle TPR = 90°$).

That is, the tetrahedron has three congruent, right isosceles triangular faces, $\triangle PRS, \triangle PST$, and $\triangle PTR$.
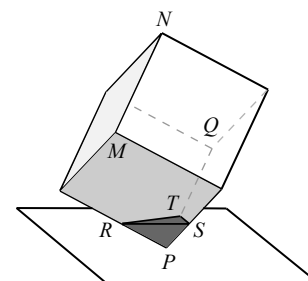
Since these three triangles are congruent, then the sides RS, ST and TR are all equal in length and thus the fourth face, $\triangle RST$, is equilateral.

In the second diagram, the tetrahedron has been repositioned to sit on $\triangle PRT$. In this position, we call the base of the tetrahedron $\triangle PRT$ and thus the height of the tetrahedron is PS since PS is perpendicular to $\triangle PRT$ (PS is perpendicular to both PR and PT).
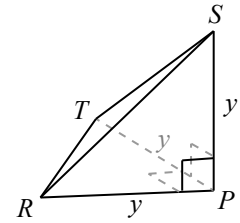
Since the tetrahedron is a triangular-based pyramid, its volume is $\frac{1}{3}|\triangle PRT| \times PS$ (where $|\triangle PRT|$ denotes the area of $\triangle PRT$).

Suppose that $PR = PS = PT = y$, as shown. (Recall that these are all equal in length by symmetry.)

In $\triangle PRT$, PR and PT are perpendicular and so $|\triangle PRT| = \frac{1}{2}PR \times PT = \frac{1}{2}y^2$.

Therefore, the volume of the tetrahedron is $\frac{1}{3}|\triangle PRT| \times PS = \frac{1}{3} \times \frac{1}{2}y^2 \times y = \frac{1}{6}y^3$.

In part (a) we found the volume of water to be equal to 81 cm^3, and since no water has been lost, the volume is the same in this new orientation of the cube.

That is, $\frac{1}{6}y^3 = 81$ or $y^3 = 486$ and so $y = \sqrt[3]{486}$.

In the third diagram shown, the tetrahedron has been repositioned again so that it is easier to visualize its vertical height, $PF = h$, the length that we are asked to find.
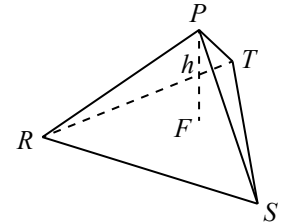
Since opposite corner N is directly above corner P, the line segment PN is perpendicular to the ground.

We call the point of intersection of PN and the top surface of the water, point F.

Using the Pythagorean Theorem in $\triangle PRS$, we get $RS^2 = PR^2 + PS^2 = y^2 + y^2 = 2y^2$.

Since $RS > 0$, then $RS = \sqrt{2}y$ and so $RS = ST = TR = \sqrt{2}y$.

(We could have used the fact that $\triangle PRS$ is a special $45° - 45° - 90°$ triangle and so $PR : PS : RS = 1 : 1 : \sqrt{2}$.)

Again, since the tetrahedron is a triangular-based pyramid its volume is $\frac{1}{3}|\triangle RST| \times h$.

We first determine the area of $\triangle RST$ by constructing altitude RM, as shown in the fourth diagram.

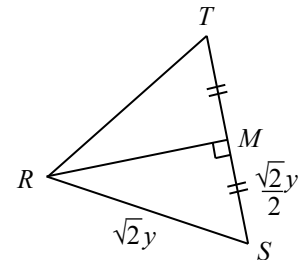Point M is the midpoint of TS and thus $MS = \frac{\sqrt{2}}{2}y$.

By the Pythagorean Theorem, $RS^2 = RM^2 + MS^2$

or $RM^2 = (\sqrt{2}y)^2 - \left(\frac{\sqrt{2}}{2}y\right)^2$.

So $RM^2 = 2y^2 - \frac{1}{2}y^2 = \frac{3}{2}y^2$, and therefore $RM = \sqrt{\frac{3}{2}}y$ (since $RM > 0$).

(We could have used the fact that $\triangle RMS$ is a special $30° - 60° - 90°$ triangle and so $MS : SR : RM = 1 : 2 : \sqrt{3}$.)

The area of $\triangle RST$ is then $\frac{1}{2} \times TS \times RM = \frac{1}{2} \times \sqrt{2}y \times \sqrt{\frac{3}{2}}y = \frac{\sqrt{3}}{2}y^2$, and so the volume of the tetrahedron (the volume of the water) is $\frac{1}{3}\left(\frac{\sqrt{3}}{2}y^2\right)h$ or $\frac{\sqrt{3}}{6}y^2h$.

Finally we substitute $y = \sqrt[3]{486}$ into $\frac{\sqrt{3}}{6}y^2h$, our expression for the volume of the tetrahedron, so that $\frac{\sqrt{3}}{6}y^2h$ becomes $\frac{\sqrt{3}}{6}(\sqrt[3]{486})^2h$.

Again, in part (a) we found the volume of water to be equal to 81 cm^3, and since no water has been lost, the volume is the same in this new orientation of the cube.

That is, $\frac{\sqrt{3}}{6}(\sqrt[3]{486})^2h = 81$ or $h = \dfrac{6 \times 81}{\sqrt{3}(\sqrt[3]{486})^2} = \dfrac{486}{\sqrt{3} \times 486^{\frac{2}{3}}} = \dfrac{486^{\frac{1}{3}}}{\sqrt{3}} \approx 4.539$.

To the nearest hundredth of a centimetre, the depth of water in the cube is 4.54 cm.