The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Cayley Contest

(Grade 10)

Thursday, February 21, 2013
(in North America and South America)

Friday, February 22, 2013
(outside of North America and South America)

Solutions

1. Simplifying, $\dfrac{8+4}{8-4} = \dfrac{12}{4} = 3$.

   Answer: (B)

2. Since $2^1 = 2$ and $2^2 = 2 \times 2 = 4$ and $2^3 = 2 \times 2 \times 2 = 8$, then $2^3 + 2^2 + 2^1 = 8 + 4 + 2 = 14$.

   Answer: (C)

3. If $x + \sqrt{81} = 25$, then $x + 9 = 25$ or $x = 16$.

   Answer: (A)

4. We could use a calculator to divide each of the four given numbers by 3 to see which calculations give an integer answer.
   Alternatively, we could use the fact that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.
   The sums of the digits of 222, 2222, 22 222, and 222 222 are 6, 8, 10, and 12, respectively.
   Two of these sums are divisible by 3 (namely, 6 and 12) so two of the four integers (namely, 222 and 222 222) are divisible by 3.

   Answer: (C)

5. Since the field originally has length 20 m and width 5 m, then its area is $20 \times 5 = 100$ m^2.
   The new length of the field is $20 + 10 = 30$ m, so the new area is $30 \times 5 = 150$ m^2.
   The increase in area is $150 - 100 = 50$ m^2.
   (Alternatively, we could note that since the length increases by 10 m and the width stays constant at 5 m, then the increase in area is $10 \times 5 = 50$ m^2.)

   Answer: (C)

6. Since the tick marks divide the cylinder into four parts of equal volume, then the level of the milk shown is a bit less than $\frac{3}{4}$ of the total volume of the cylinder.
   Three-quarters of the total volume of the cylinder is $\frac{3}{4} \times 50 = 37.5$ L.
   Of the five given choices, the one that is slightly less than 37.5 L is 36 L, or (D).

   Answer: (D)

7. Since $\triangle PQR$ is equilateral, then $PQ = QR = RP$.
   Therefore, $4x = x + 12$ or $3x = 12$ and so $x = 4$.

   Answer: (C)

8. Using the definition of the symbol, $3 \diamond 6 = \dfrac{3+6}{3 \times 6} = \dfrac{9}{18} = \dfrac{1}{2}$.

   Answer: (E)

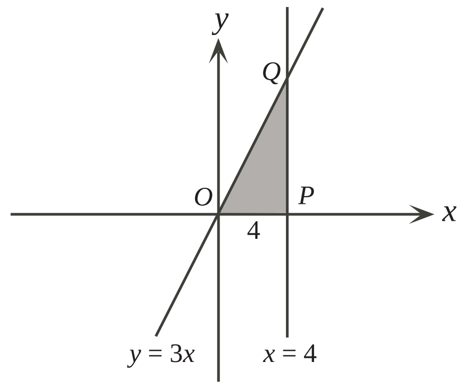
9. One way to phrase the Pythagorean Theorem is that the area of the square formed on the hypotenuse of a right-angled triangle equals the sum of the areas of the squares formed on the other two sides.
   Therefore, the area of the square on PQ equals the area of the square on PR minus the area of the square on QR, which equals $169 - 144$ or 25.

   Answer: (E)

10. Since the average age of the three sisters is 27, then the sum of their ages is $3 \times 27 = 81$.
    When Barry is included the average age of the four people is 28, so the sum of the ages of the four people is $4 \times 28 = 112$.
    Barry's age is the difference between the sum of the ages of all four people and the sum of the ages of the three sisters, which equals $112 - 81$ or 31.

    Answer: (E)

11. Let O be the origin (where the line with equation $y = 3x$ intersects the x-axis).
Let P be the point where the line with equation $x = 4$ intersects the x-axis, and let Q be the point where the lines with equations $x = 4$ and $y = 3x$ intersect.



The line $x = 4$ is perpendicular to the x-axis, so the given triangle is right-angled at P.
Therefore, the area of the triangle equals $\frac{1}{2}(OP)(PQ)$.
Now P lies on the x-axis and on the line $x = 4$, so has coordinates $(4, 0)$.
Thus, $OP = 4$.
Point Q also has x-coordinate 4. Since Q lies on $y = 3x$, then its y-coordinate is $y = 3(4) = 12$.
Since P has coordinates $(4, 0)$ and Q has coordinates $(4, 12)$, then $PQ = 12$.
Therefore, the area of the triangle is $\frac{1}{2}(4)(12) = 24$.

Answer: (B)

12. *Solution 1*
Since $a(x + b) = 3x + 12$ for all x, then $ax + ab = 3x + 12$ for all x.
Since the equation is true for all x, then the coefficients on the left side must match the coefficients on the right side.
Therefore, $a = 3$ and $ab = 12$, which gives $3b = 12$ or $b = 4$.
Finally, $a + b = 3 + 4 = 7$.

Solution 2
Since $a(x + b) = 3x + 12$ for all x, then the equation is true for $x = 0$ and $x = 1$.
When $x = 0$, we obtain $a(0 + b) = 3(0) + 12$ or $ab = 12$.
When $x = 1$, we obtain $a(1 + b) = 3(1) + 12$ or $a + ab = 15$.
Since $ab = 12$, then $a + 12 = 15$ or $a = 3$.
Since $ab = 12$ and $a = 3$, then $b = 4$.
Finally, $a + b = 3 + 4 = 7$.

Answer: (D)

13. *Solution 1*
If $x = 1$, then $3x + 1 = 4$, which is an even integer.
In this case, the four given choices are

(A) $x + 3 = 4$ (B) $x - 3 = -2$ (C) $2x = 2$ (D) $7x + 4 = 11$ (E) $5x + 3 = 8$

Of these, the only odd integer is (D). Therefore, (D) must be the correct answer as the result must be true no matter what integer value of x is chosen that makes $3x + 1$ even.

Solution 2

If x is an integer for which $3x+1$ is even, then $3x$ is odd, since it is 1 less than an even integer.
If $3x$ is odd, then x must be odd (since if x is even, then $3x$ would be even).
If x is odd, then $x+3$ is even (odd plus odd equals even), so (A) cannot be correct.
If x is odd, then $x-3$ is even (odd minus odd equals even), so (B) cannot be correct.
If x is odd, then $2x$ is even (even times odd equals even), so (C) cannot be correct.
If x is odd, then $7x$ is odd (odd times odd equals odd) and so $7x+4$ is odd (odd plus even equals odd).
If x is odd, then $5x$ is odd (odd times odd equals odd) and so $5x+3$ is even (odd plus odd equals even), so (E) cannot be correct.
Therefore, the one expression which must be odd is $7x+4$.

Answer: (D)

14. With a given set of four digits, the largest possible integer that can be formed puts the largest digit in the thousands place, the second largest digit in the hundreds place, the third largest digit in the tens place, and the smallest digit in the units place. This is because the largest digit can make the largest contribution in the place with the most value.
Thus, the largest integer that can be formed with the digits 2, 0, 1, 3 is 3210.
With a given set of digits, the smallest possible integer comes from listing the numbers in increasing order from the thousands place to the units place.
Here, there is an added wrinkle that the integer must be at least 1000. Therefore, the thousands digit is at least 1. The smallest integer of this type that can be made uses a thousands digit of 1, and then lists the remaining digits in increasing order; this integer is 1023.
The difference between these integers is $3210-1023=2187$.

Answer: (A)

15. Since 40% of the songs on the updated playlist are Country, then the remaining $100\%-40\%$ or 60% must be Hip Hop and Pop songs.
Since the ratio of Hip Hop song to Pop songs does not change, then 65% of this remaining 60% must be Hip Hop songs.
Overall, this is $65\% \times 60\% = 0.65 \times 0.6 = 0.39 = 39\%$ of the total number of songs on the playlist.

Answer: (E)

16. First, we note that $5^{35}-6^{21}$ is a positive integer, since

$$5^{35}-6^{21}=(5^5)^7-(6^3)^7=3125^7-216^7$$

and $3125>216$.
Second, we note that any positive integer power of 5 has a units digit of 5. Since $5\times 5=25$ and this product has a units digit of 5, then the units digit of 5^3 is obtained by multiplying 5 by the units digit 5 of 25. Thus, the units digit of 5^3 is 5. Similarly, each successive power of 5 has a units digit of 5.
Similarly, each power of 6 has a units digit of 6.
Therefore, 5^{35} has a units digit of 5 and 6^{21} has a units digit of 6. When a positive integer with units digit 6 is subtracted from a larger positive integer whose units digit is 5, the difference has a units digit of 9.
Therefore, $5^{35}-6^{21}$ has a units digit of 9.

Answer: (B)

17. We have

$$\begin{aligned}
\text{Perimeter of } \triangle PST &= PS + ST + PT \\
&= PS + (SU + UT) + PT \\
&= PS + SQ + TR + PT \qquad \text{(since } SU = SQ \text{ and } UT = TR\text{)} \\
&= (PS + SQ) + (PT + TR) \\
&= PQ + PR \\
&= 19 + 17 \\
&= 36
\end{aligned}$$

Therefore, the perimeter of $\triangle PQR$ is 36.

ANSWER: (A)

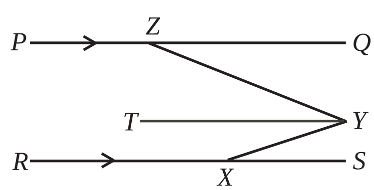
18. Suppose that the quotient of the division of 109 by x is q.
Since the remainder is 4, this is equivalent to $109 = qx + 4$ or $qx = 105$.
Put another way, x must be a positive integer divisor of 105.
Since $105 = 5 \times 21 = 5 \times 3 \times 7$, its positive integer divisors are

$$1, 3, 5, 7, 15, 21, 35, 105$$

Of these, 15, 21 and 35 are two-digit positive integers so are the possible values of x.
The sum of these values is $15 + 21 + 35 = 71$.

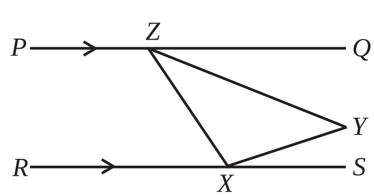
ANSWER: (D)

19. *Solution 1*
Draw a line segment TY through Y parallel to PQ and RS, as shown, and delete line segment ZX.



Since TY is parallel to RS, then $\angle TYX = \angle YXS = 20^\circ$.
Thus, $\angle ZYT = \angle ZYX - \angle TYX = 50^\circ - 20^\circ = 30^\circ$.
Since PQ is parallel to TY, then $\angle QZY = \angle ZYT = 30^\circ$.

Solution 2
Since QZ and XS are parallel, then $\angle QZX + \angle ZXS = 180^\circ$.



Now $\angle QZX = \angle QZY + \angle YZX$ and $\angle ZXS = \angle ZXY + \angle YXS$.
We know that $\angle YXS = 20^\circ$.
Also, the sum of the angles in $\triangle XYZ$ is 180°, so $\angle YZX + \angle ZXY + \angle ZYX = 180^\circ$, or $\angle YZX + \angle ZXY = 180^\circ - \angle ZYX = 180^\circ - 50^\circ = 130^\circ$.
Combining all of these facts, we obtain $\angle QZY + \angle YZX + \angle ZXY + \angle YXS = 180^\circ$ or $\angle QZY + 130^\circ + 20^\circ = 180^\circ$.
From this, we obtain $\angle QZY = 180^\circ - 130^\circ - 20^\circ = 30^\circ$.

ANSWER: (A)

20. Suppose that the length of the route is d km.
Then Jill jogs $\frac{d}{2}$ km at 6 km/h and runs $\frac{d}{2}$ km at 12 km/h.
Note that time equals distance divided by speed.
Since her total time was x hours, then $x = \frac{d/2}{6} + \frac{d/2}{12} = \frac{d}{12} + \frac{d}{24} = \frac{2d}{24} + \frac{d}{24} = \frac{3d}{24} = \frac{d}{8}$.
Also, Jack walks $\frac{d}{3}$ km at 5 km/h and runs $\frac{2d}{3}$ km at 15 km/h.
Since his total time is y hours, then $y = \frac{d/3}{5} + \frac{2d/3}{15} = \frac{d}{15} + \frac{2d}{45} = \frac{3d}{45} + \frac{2d}{45} = \frac{5d}{45} = \frac{d}{9}$.
Finally, $\frac{x}{y} = \frac{d/8}{d/9} = \frac{9}{8}$.

ANSWER: (A)

21. We start by analyzing the given sum as if we were performing the addition by hand. Doing this, we would start with the units column.
Here, we see that the units digit of the sum $X + Y + Z$ is X.
Thus, the units digit of $Y + Z$ must be 0.
Because none of the digits is zero, then there must be a carry to the tens column.
Because $Y + Z$ are different digits between 1 and 9, then $Y + Z$ is at most 17. Since the units digit of $Y + Z$ is 0, then $Y + Z = 10$ and there is a carry of 1 to the tens column.
Comparing the sums and digits in the tens and units columns, we see that $Y = X + 1$ (since Y cannot be 0).

$$\begin{array}{cccc} & & {\scriptstyle 1} & \\ & X & X & X \\ & Y & Y & Y \\ + & Z & Z & Z \\ \hline Z & Y & Y & X \end{array}$$

Here are two ways that we could finish the solution.

<u>Method 1</u>
Since $Y + Z = 10$, then $YYY + ZZZ = 1110$.
In other words, each column gives a carry of 1 that is added to the next column to the left.
Alternatively, we could notice that $YYY = Y \times 111$ and $ZZZ = Z \times 111$.
Thus, $YYY + ZZZ = (Y + Z) \times 111 = 10 \times 111 = 1110$.
Therefore, the given sum simplifies to $1110 + XXX = ZYYX$.
If $X = 9$, then the sum would be $1110 + 999 = 2009$, which has $Y = 0$, which can't be the case.
Therefore, $X \leq 8$.
This tells us that $1110 + XXX$ is at most $1110 + 888 = 1998$, and so Z must equal 1, regardless of the value of X.
Since $Y + Z = 10$, then $Y = 9$.
This means that we have $1110 + XXX = 199X$.
Since X is a digit, then $X + 1 = 9$, and so $X = 8$, which is consistent with the above.
(Checking, if $X = 8$, $Y = 9$, $Z = 1$, we have $888 + 999 + 111 = 1998$.)

Method 2

Consider the tens column.

The sum in this column is $1 + X + Y + Z$ (we add 1 as the "carry" from the units column).

Since $Y + Z = 10$, then $1 + X + Y + Z = 11 + X$.

Since X is between 1 and 9, then $11 + X$ is at least 12 and at most 20.

In fact, $11 + X$ cannot equal 20, since this would mean entering a 0 in the sum, and we know that none of the digits is 0.

$$\begin{array}{cccc} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & X & X & X \\ & Y & Y & Y \\ + & Z & Z & Z \\ \hline 1 & Y & Y & X \end{array}$$

Thus, $11 + X$ is less than 20, and so must equal $10 + Y$ (giving a digit of Y in the tens column of the sum and a carry of 1 to the hundreds column).

Further, since the tens column and the hundreds column are the same, then the carry to the thousands column is also 1. In other words, $Z = 1$.

Since $Y + Z = 10$ and $Z = 1$, then $Y = 9$.

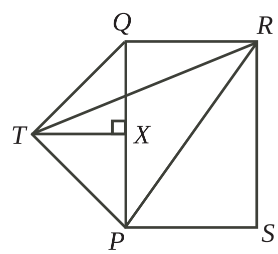
Since $Y = X + 1$ and $Y = 9$, then $X = 8$.

As in Method 1, we can check that these values satisfy the given sum.

Answer: (D)

22. *Solution 1*

Let X be the point on QP so that TX is perpendicular to QP.



Since $\triangle QTP$ is isosceles, then X is the midpoint of QP.

Since $QP = 4$, then $QX = XP = 2$.

Since $\angle TQP = 45^\circ$ and $\angle QXT = 90^\circ$, then $\triangle QXT$ is also isosceles and right-angled.

Therefore, $TX = QX = 2$.

We calculate the area of $\triangle PTR$ by adding the areas of $\triangle QRP$ and $\triangle QTP$ and subtracting the area of $\triangle QRT$.

Since $QR = 3$, $PQ = 4$ and $\angle PQR = 90^\circ$, then the area of $\triangle QRP$ is $\frac{1}{2}(3)(4) = 6$.

Since $QP = 4$, $TX = 2$ and TX is perpendicular to QP, then the area of $\triangle QTP$ is $\frac{1}{2}(4)(2) = 4$.

We can view $\triangle QRT$ as having base QR with its height being the perpendicular distance from QR to T, which equals the length of QX. Thus, the area of $\triangle QRT$ is $\frac{1}{2}(3)(2) = 3$.

Therefore, the area of $\triangle PTR$ is $6 + 4 - 3 = 7$.

Solution 2

Let X be the point on QP so that TX is perpendicular to QP.

Since $\triangle QTP$ is isosceles, then X is the midpoint of QP.

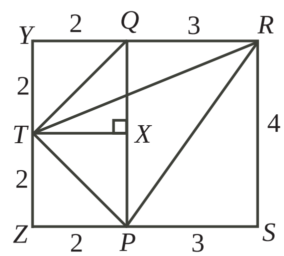
Since $QP = 4$, then $QX = XP = 2$.

Since $\angle TQP = 45^\circ$ and $\angle QXT = 90^\circ$, then $\triangle QXT$ is also isosceles and right-angled.

Therefore, $TX = QX = 2$.

Extend RQ to Y and SP to Z so that YZ is perpendicular to each of YR and ZS and so that YZ passes through T.

Each of $YQXT$ and $TXPZ$ has three right angles (at Y, Q and X, and X, P and Z, respectively), so each of these is a rectangle.
Since $QX = TX = XP = 2$, then each of $YQXT$ and $TXPZ$ is a square with side length 2.
Now $YRSZ$ is a rectangle with $YR = YQ + QR = 2 + 3 = 5$ and $RS = 4$.



The area of $\triangle PTR$ equals the area of rectangle $YRSZ$ minus the areas of $\triangle TYR$, $\triangle RSP$ and $\triangle PZT$.
Rectangle $YRSZ$ is 5 by 4 and so has area $5 \times 4 = 20$.
Since $TY = 2$ and $YR = 5$ and TY is perpendicular to YR, then the area of $\triangle TYR$ is $\frac{1}{2}(TY)(YR) = 5$.
Since $RS = 4$ and $SP = 3$ and RS is perpendicular to SP, then the area of $\triangle RSP$ is $\frac{1}{2}(RS)(SP) = 6$.
Since $PZ = ZT = 2$ and PZ is perpendicular to ZT, then the area of $\triangle PZT$ is $\frac{1}{2}(PZ)(ZT) = 2$.
Therefore, the area of $\triangle PTR$ is $20 - 5 - 6 - 2 = 7$.

Answer: (C)

23. First, we consider the first bag, which contains a total of $2 + 2 = 4$ marbles.
There are 4 possible marbles that can be drawn first, leaving 3 possible marbles that can be drawn second. This gives a total of $4 \times 3 = 12$ ways of drawing two marbles.
For both marbles to be red, there are 2 possible marbles (either red marble) that can be drawn first, and 1 marble that must be drawn second (the remaining red marble). This gives a total of $2 \times 1 = 2$ ways of drawing two red marbles.
For both marbles to be blue, there are 2 possible marbles that can be drawn first, and 1 marble that must be drawn second. This gives a total of $2 \times 1 = 2$ ways of drawing two blue marbles.
Therefore, the probability of drawing two marbles of the same colour from the first bag is the total number of ways of drawing two marbles of the same colour ($2 + 2 = 4$) divided by the total number of ways of drawing two marbles (12), or $\dfrac{4}{12} = \dfrac{1}{3}$.
Second, we consider the second bag, which contains a total of $2 + 2 + g = g + 4$ marbles.
There are $g + 4$ possible marbles that can be drawn first, leaving $g + 3$ possible marbles that can be drawn second. This gives a total of $(g + 4)(g + 3)$ ways of drawing two marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two red marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two blue marbles.
For both marbles to be green, there are g possible marbles that can be drawn first, and $g - 1$ marbles that must be drawn second. This gives a total of $g(g - 1)$ ways of drawing two green marbles.
Therefore, the probability of drawing two marbles of the same colour from the second bag is the total number of ways of drawing two marbles of the same colour ($2 + 2 + g(g - 1) = g^2 - g + 4$) divided by the total number of ways of drawing two marbles ($(g + 4)(g + 3)$), or $\dfrac{g^2 - g + 4}{(g + 4)(g + 3)}$.

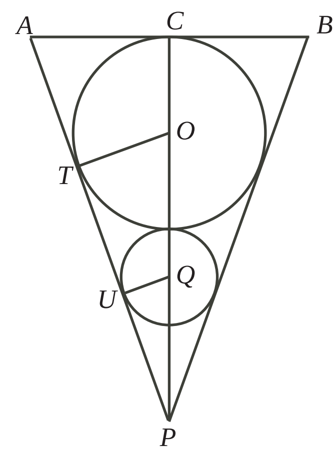
Since the two probabilities that we have calculated are to be equal and $g \neq 0$, then

$$\begin{aligned}
\frac{1}{3} &= \frac{g^2 - g + 4}{(g+4)(g+3)} \\
(g+4)(g+3) &= 3g^2 - 3g + 12 \\
g^2 + 7g + 12 &= 3g^2 - 3g + 12 \\
10g &= 2g^2 \\
g &= 5 \qquad (\text{since } g \neq 0)
\end{aligned}$$

Therefore, $g = 5$.

ANSWER: (B)

24. Let the radius of the smaller sphere be r.
Thus, the radius of the larger sphere is $2r$.
We determine expressions for the height and radius of the cone in terms of r and use these to help solve the problem.
By symmetry, the centres of the two spheres (Q of the smaller sphere and O of the larger sphere) lie on the line joining the centre of the circular top of the cone (C) to the tip of the cone (P).
Draw a vertical cross-section of the cone through the centre of the circular top of the cone and through the tip of the cone.
Each such cross-section will be an identical triangle.
Because the centres of the spheres lie on a line which is in the plane of this cross-section, the cross-section of each sphere will be a "great" circle (that is, the largest possible circular cross-section of the sphere).
Because the top of the larger sphere is just level with the top of the cone, then the sphere "touches" the circular top of the cone at its centre C.
Finally, because the spheres touch the cone all the way around, then the circles will be tangent to the triangle in the cross-section.



We label the triangular cross section as ABP.
Note that CP is perpendicular to AB at C.
Draw radii from O and Q to the points T and U, respectively, on AP where the circles with centre O and Q are tangent to AP.
Note that OT and QU are perpendicular to AP, with $OT = 2r$ and $QU = r$.
Also, since the two circles are just touching, then the line segment joining their centres, OQ, passes through this point of tangency, and so $OQ = 2r + r = 3r$.
Now $\triangle OTP$ is similar to $\triangle QUP$, since each is right-angled and they share a common angle at P.
Since $\dfrac{OT}{QU} = \dfrac{2r}{r} = 2$, then $\dfrac{OP}{QP} = 2$ or $OP = 2QP$.
Since $OP = OQ + QP = 3r + QP$, then $3r + QP = 2QP$ or $QP = 3r$.
Thus, the height of the cone is $CP = CO + OQ + QP = 2r + 3r + 3r = 8r$.
(Note that CO is a radius of the larger circle, so $CO = 2r$.)
We also see that $\triangle ACP$ is similar to $\triangle QUP$, since $\triangle ACP$ is also right-angled (at C) and shares the angle at P.
Thus, $\dfrac{AC}{CP} = \dfrac{QU}{UP}$.

We know that $CP = 8r$ and $QU = r$.
To calculate UP, we use the Pythagorean Theorem to get

$$UP = \sqrt{QP^2 - QU^2} = \sqrt{(3r)^2 - r^2} = \sqrt{8r^2} = \sqrt{8}r$$

since $r > 0$.
Therefore, $AC = \dfrac{CP \cdot QU}{UP} = \dfrac{8r \cdot r}{\sqrt{8}r} = \sqrt{8}r$.
Finally, we can use the given information. We are told that the volume of water remaining after the full cone has the two spheres added is 2016π. This is equivalent to saying that the difference between the volume of the cone and the combined volumes of the spheres is 2016π.
Using the given volume formulae, we obtain

$$\begin{aligned}
\tfrac{1}{3}\pi(AC)^2(CP) - \tfrac{4}{3}\pi(QU)^3 - \tfrac{4}{3}\pi(OT)^3 &= 2016\pi \\
\tfrac{1}{3}\pi(\sqrt{8}r)^2(8r) - \tfrac{4}{3}\pi r^3 - \tfrac{4}{3}\pi(2r)^3 &= 2016\pi \\
64\pi r^3 - 4\pi r^3 - 32\pi r^3 &= 6048\pi \\
28r^3 &= 6048 \\
r^3 &= 216 \\
r &= 6
\end{aligned}$$

Therefore, the radius of the smaller sphere is 6.

Answer: (B)

25. We define $L(n) = n - Z(n!)$ to be the nth number in Lloyd's list.
We note that the number of trailing zeros in any positive integer m (which is $Z(m)$) equals the number of factors of 10 that m has. For example, 2400 has two trailing zeros since $2400 = 24 \times 10 \times 10$. Further, since $10 = 2 \times 5$, the number of factors of 10 in any positive integer m is determined by the number of factors of 2 and 5.
Consider $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.
Since $5 > 2$, then $n!$ will always contain more factors of 2 than factors of 5. This is because if we make a list of the multiples of 2 and a list of the multiples of 5, then there will be more numbers in the first list than in the second list that are less than or equal to a given positive integer n (and so numbers in the first list that contribute more than one factor of 2 will occur before numbers in the second list that contribute more than one factor of 5, and so on).
In other words, the value of $Z(n!)$ will equal the number of factors of 5 that $n!$ has.
We use the notation $V(m)$ to represent the number of factors of 5 in the integer m.
Thus, $Z(n!) = V(n!)$ and so $L(n) = n - V(n!)$.
Since $(n+1)! = (n+1) \times n!$, then $V((n+1)!) = V(n+1) + V(n!)$. (This is because any additional factors of 5 in $(n+1)!$ that are not in $n!$ come from $n+1$.)
Therefore, if $n+1$ is not a multiple of 5, then $V(n+1) = 0$ and so $V((n+1)!) = V(n!)$.
If $n+1$ is a multiple of 5, then $V(n+1) > 0$ and so $V((n+1)!) > V(n!)$.
Note that

$$\begin{aligned}
L(n+1) - L(n) &= ((n+1) - V((n+1)!)) - (n - V(n!)) \\
&= ((n+1) - n) - (V((n+1)!) - V(n!)) \\
&= 1 - V(n+1)
\end{aligned}$$

If $n+1$ is not a multiple of 5, then $V(n+1) = 0$ and so $L(n+1) - L(n) = 1$.
This tells us that when $n+1$ is not a multiple of 5, the corresponding term in the list is one

larger than the previous term; thus, the terms in the list increase by 1 for four terms in a row whenever there is not a multiple of 5 in this list (since multiples of 5 occur every fifth integer). When $n+1$ is a multiple of 5, the corresponding term will be the same as the previous one (if $n+1$ includes only one factor of 5) or will be smaller if $n+1$ includes more than one factor of 5.
After a bit of experimentation, it begins to appear that, in order to get an integer to appear three times in the list, there needs to be an integer n that contains at least five factors of 5.
We explicitly show that there are six integers that appear three times in the list $L(100)$ to $L(10\,000)$. Since 6 is the largest of the answer choices, then 6 must be the correct answer.

Let $N = 5^5 k = 3125k$ for some positive integer k. If $N \leq 10\,000$, then k can equal 1, 2 or 3. Also, define $a = L(N)$.
We make a table of the values of $L(N-6)$ to $L(N+6)$. We note that since N contains five factors of 5, then $N-5$ and $N+5$ are each divisible by 5 (containing only one factor of 5 each), and none of the other integers in the list is divisible by 5. Also, we note that $L(m+1) - L(m) = 1 - V(m+1)$ as seen above.

m	$N-6$	$N-5$	$N-4$	$N-3$	$N-2$	$N-1$	N	$N+1$	$N+2$	$N+3$	$N+4$	$N+5$	$N+6$
$V(m)$	0	1	0	0	0	0	5	0	0	0	0	1	0
$L(m)$	a	a	$a+1$	$a+2$	$a+3$	$a+4$	a	$a+1$	$a+2$	$a+3$	$a+4$	$a+4$	$a+5$

Therefore, if $N = 5^5 k$, then the integers $L(N) = a$ and $L(N) + 4 = a + 4$ each appear in the list three times.
Since there are three values of k that place N in the range $100 \leq N \leq 10\,000$, then there are six integers in Lloyd's list that appear at least three times.

In order to prove that there are no other integers that appear at least three times in the list (rather than relying on the multiple choice nature of the problem), we would need to prove some additional facts. One way to do this would be to prove:

> If n and k are positive integers with $k \geq 7$ and $n \leq 10\,000$ and $n + k \leq 10\,000$, then $L(n+k) \neq L(n)$.

This allows us to say that if two integers in the list are equal, then they must come from $L(n)$ to $L(n+6)$ inclusive, for some n. This then would allow us to say that if three integers in the list are equal, then they must come from $L(n-6)$ to $L(n+6)$, inclusive, for some n. Finally, we could then prove:

> If n is a positive integer with $n \leq 10\,000$ with three of the integers from $L(n-6)$ to $L(n+6)$, inclusive, equal, then one of the integers in the list $n-6$ to $n+6$ must be divisible by 3125.

These facts together allow us to reach the desired conclusion.

ANSWER: (E)