The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Canadian Intermediate Mathematics Contest

Thursday, November 21, 2013
(in North America and South America)

Friday, November 22, 2013
(outside of North America and South America)

Solutions

Part A

1. Since ABC is a straight line segment, then $\angle CBE = 180^\circ - \angle ABE = 180^\circ - 130^\circ = 50^\circ$.
   Since the angles in $\triangle BCE$ add to 180°, then

   $$\angle BCE = 180^\circ - \angle CBE - \angle BEC = 180^\circ - 50^\circ - 60^\circ = 70^\circ$$

   Since BCD is a straight line segment, then $\angle DCE = 180^\circ - \angle BCE = 180^\circ - 70^\circ = 110^\circ$.
   Therefore, $x^\circ = 110^\circ$ and so $x = 110$.
   (Alternatively, we could have noted that $\angle DCE$ is an exterior angle of $\triangle BCE$.
   Thus, $\angle DCE = \angle CBE + \angle BEC = 50^\circ + 60^\circ = 110^\circ$, which also gives $x = 110$.)

   Answer: 110

2. *Solution 1*
   Any positive integer that is a multiple of 8 is also a multiple of each of 2 and 4, because 8 itself is a multiple of 2 and 4.
   Therefore, we are looking for the smallest positive integer that is a multiple of 6 and 8.
   We consider positive multiples of 8, starting with the smallest multiples, and determine the smallest one that is also a multiple of 6.
   $1 \times 8 = 8$ is not a multiple of 6.
   $2 \times 8 = 16$ is not a multiple of 6.
   $3 \times 8 = 24$ is a multiple of 6, and so is the smallest positive integer that is a multiple of each of 2, 4, 6, and 8.

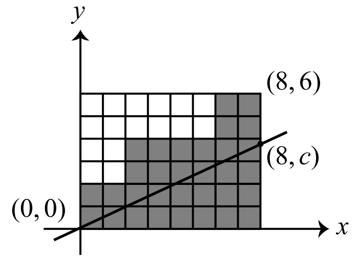
   Solution 2
   Any positive integer that is a multiple of 8 is also a multiple of each of 2 and 4, because 8 itself is a multiple of 2 and 4.
   Therefore, we are looking for the smallest positive integer that is a multiple of 6 and 8.
   This integer is the least common multiple of 6 and 8.
   We note that $6 = 2 \times 3$ and $8 = 2^3$.
   The least common multiple of 6 and 8 can be obtained by first looking at the prime factors of these integers. The only prime factors of 6 and 8 are 2 and 3. The factor 3 occurs a maximum of 1 time (in 6) and the factor 2 occurs a maximum of 3 times (in 8).
   Thus, the least common multiple of 6 and 8 is $3 \times 2^3 = 24$.
   Therefore, the smallest positive integer that is a multiple of each of 2, 4, 6, and 8 is 24.

   Answer: 24

3. Since $x = 3$ and $y = 7$ satisfy the given relation $y = ax + (1 - a)$, then $7 = 3a + (1 - a)$ or $7 = 2a + 1$.
   Thus, $2a = 6$ and so $a = 3$.
   When $x = 8$, we therefore obtain $y = ax + (1 - a) = 3(8) + (1 - 3) = 24 - 2 = 22$.

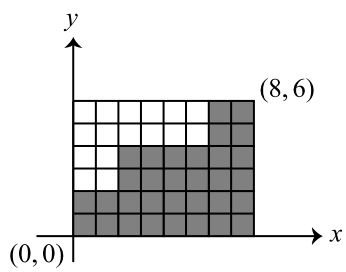
   Answer: 22

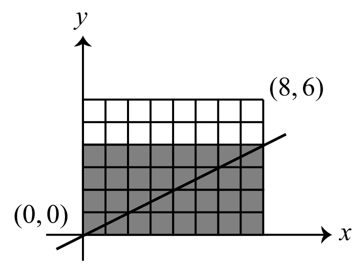
4. *Solution 1*
   Since 32 of the 1×1 squares are shaded, then the total shaded area is 32.
   We want to draw a line through $(0, 0)$ and $(8, c)$ that divides the shaded region into two pieces, each with area $\frac{32}{2} = 16$.
   As long as the slope of the line segment through $(0, 0)$ and $(8, c)$ is not too large, the bottom piece will be a triangle with vertices $(0, 0)$, $(8, 0)$, and $(8, c)$.



This triangle is right-angled at $(8,0)$ and so has base of length 8 and height c.
For the area of this triangle to be 16, we need $\frac{1}{2}(8)(c) = 16$ or $4c = 16$.
Thus, $c = 4$.
(Note that if $c = 4$, the line segment passes through $(0,0)$ and $(8,4)$. This line segment has slope $\frac{4-0}{8-0} = \frac{1}{2}$ and so passes through the points $(2,1)$ and $(4,2)$. This confirms that the line segment does not pass through any unshaded regions.)

Solution 2
We move the upper right shaded 2×2 square to complete a 8×4 shaded rectangle, as shown.





The shaded area is now a rectangle with bottom left vertex $(0,0)$ and top right vertex $(8,4)$.
The area of the rectangle is cut in half by its diagonal passing through $(0,0)$ and $(8,4)$.
Note the region affected by this move lies above this diagonal. (This is because the diagonal has slope $\frac{4-0}{8-0} = \frac{1}{2}$ and so passes through the points $(2,1)$ and $(4,2)$.)
This means that the areas of the regions below and above the line were unaffected by the move.
Thus, reversing the move does not change the areas of the regions below and above the line.
Therefore, the line drawn through $(0,0)$ and $(8,4)$ cuts the original region into two pieces of equal area.
Thus, $c = 4$.

ANSWER: $c = 4$

5. Since 1000 and 10 000 are not palindromes, then every palindrome between 1000 and 10 000 has four digits and so has the form $ABBA$ for some digits A and B with $A \neq 0$.
We are looking for palindromes $ABBA$ that are divisible by 6.
An integer is divisible by 6 whenever it is divisible by 2 (that is, it is even) and divisible by 3.
Since we want $ABBA$ to be even, then A must be even.
Therefore, $A = 2$, $A = 4$, $A = 6$, or $A = 8$.
For a positive integer to be divisible by 3, the sum of its digits must be divisible by 3.
We look at each possible value for A separately and determine values of B that give palindromes that are divisible by 6:

- Case 1: $A = 2$
  Here, $ABBA = 2BB2$ and the sum of its digits is $2 + B + B + 2 = 2B + 4$.
  We need to determine all digits B for which $2B + 4$ is divisible by 3.
  Since B is at most 9, then $2B + 4$ is at most $2(9) + 4 = 22$. Also, $2B + 4$ is even because $2B + 4 = 2(B + 2)$.
  Thus, $2B + 4$ must be an even multiple of 3 that is at most 22.
  Therefore, $2B + 4$ could equal 6, 12 or 18.
  If $2B + 4 = 6$, then $2B = 2$ or $B = 1$.
  If $2B + 4 = 12$, then $2B = 8$ or $B = 4$.
  If $2B + 4 = 18$, then $2B = 14$ or $B = 7$.
  (Alternatively, we could have checked each of the values of B from 0 to 9 individually by dividing $2BB2$ by 6.)
- Case 2: $A = 4$
  Here, $ABBA = 4BB4$.
  Using a similar method to that in Case 1, we determine that the values of B for which $4BB4$ is divisible by 6 are $B = 2, 5, 8$.
- Case 3: $A = 6$
  Here, $ABBA = 6BB6$.
  Using a similar method to that in Case 1, we determine that the values of B for which $6BB6$ is divisible by 6 are $B = 0, 3, 6, 9$.
- Case 4: $A = 8$
  Here, $ABBA = 8BB8$.
  Using a similar method to that in Case 1, we determine that the values of B for which $8BB8$ is divisible by 6 are $B = 1, 4, 7$.

Therefore, there are $3 + 3 + 4 + 3 = 13$ palindromes between 1000 and 10 000 that are multiples of 6.

ANSWER: 13

6. The given fact that $\frac{1}{6} - \frac{1}{7} = \frac{1}{6 \times 7}$ suggests that we consider $\frac{1}{n} - \frac{1}{n+1}$.
Using a common denominator,

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)} \qquad (*)$$

Starting with

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = 1$$

we subtract and add the same quantities from the left side to obtain

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} - \frac{1}{43} + \frac{1}{43} - \frac{1}{44} + \frac{1}{44} - \frac{1}{45} + \frac{1}{45} = 1$$

We then regroup the terms on the left side as

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \left(\frac{1}{42} - \frac{1}{43}\right) + \left(\frac{1}{43} - \frac{1}{44}\right) + \left(\frac{1}{44} - \frac{1}{45}\right) + \frac{1}{45} = 1$$

and use the fact $(*)$ above to obtain

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42\times 43}+\frac{1}{43\times 44}+\frac{1}{44\times 45}+\frac{1}{45}=1$$

This gives

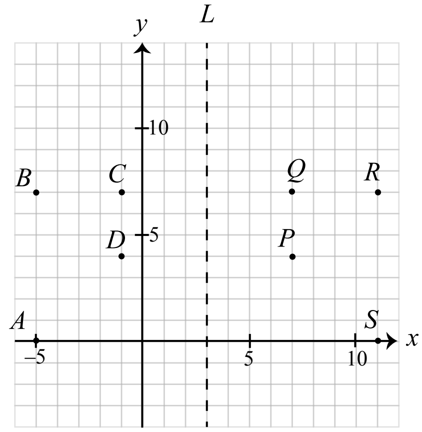
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{1806}+\frac{1}{1892}+\frac{1}{1980}+\frac{1}{45}=1$$

Therefore, a solution to the given equation that satisfies the condition $1000 < x < y < z < 2000$ is $(x, y, z) = (1806, 1892, 1980)$.
There are other solutions to this equation obeying the given conditions, but we are only asked to find one such solution.

Answer: $(1806, 1892, 1980)$

Part B

1. (a) Points D, B and A are being reflected in a vertical line. Thus, the y-coordinate of each image point will be the same as the y-coordinate of the original point.
Point D has x-coordinate -1 and is being reflected in the line $x = 3$.
Point D is thus $3 - (-1) = 4$ units to the left of the line $x = 3$.
Therefore, its image after reflection will be 4 units to the right of the line $x = 3$.
Thus, the coordinates of P are $(3 + 4, 4) = (7, 4)$.
In a similar way, we reflect point $B(-5, 7)$ in the line $x = 3$ to obtain the point R whose coordinates are $(3 + 8, 7) = (11, 7)$.
Finally, since the x-coordinate of A is the same as the x-coordinate of B, then the x-coordinate of S is the same as the x-coordinate of R.
Thus, the coordinates of S are $(11, 0)$.



(b) The perimeter of figure $ABCDPQRS$ equals the sum of the lengths of its 8 sides. Each side is either vertical or horizontal. The length of each horizontal side can be found by subtracting x-coordinates; the length of each vertical side can be found by subtracting y-coordinates.
The perimeter equals $AB + BC + CD + DP + PQ + QR + RS + SA$.
Given coordinates $A(-5, 0)$ and $B(-5, 7)$, we obtain $AB = 7 - 0 = 7$.
Given coordinates $B(-5, 7)$ and $C(-1, 7)$, we obtain $BC = -1 - (-5) = 4$.
Similarly, $CD = 3$, $DP = 8$, $PQ = 3$, $QR = 4$, $RS = 7$, and $SA = 16$.
Therefore, the perimeter of $ABCDPQRS$ is $7 + 4 + 3 + 8 + 3 + 4 + 7 + 16 = 52$.
(Alternatively, the perimeter of the figure can be found by calculating the perimeter of the figure to the left of line L and then doubling this total. This is because the figure is formed by reflecting the portion to the left of line L to create the entire figure.)

(c) The larger cylinder has height equal to the length of AB, which is 7, from part (b).
The smaller cylinder has height equal to the length of CD, which is 3 from part (b).
The radius of the larger cylinder equals the distance from L to each of B, R, S, and A. In part (a), we saw that this distance is 8.
The radius of the smaller cylinder equals the distance from L to each of C, Q, P, and D. In part (a), we saw that this distance is 4.

The volume of the given solid equals the volume of the larger cylinder minus the volume of the smaller cylinder.
Since the volume of a cylinder with radius r and height h is $\pi r^2 h$, then the volume of the solid equals

$$\pi(8^2)(7) - \pi(4^2)(3) = 448\pi - 48\pi = 400\pi$$

The surface area of the given solid consists of the bottom and lateral faces of the larger cylinder (the outside), the bottom and lateral faces of the smaller cylinder (inside the hole), and the top face of the larger cylinder with the top face of the smaller cylinder removed.
Since the surface area of a cylinder with radius r and height h is $2\pi r^2 + 2\pi rh$, then the surface area of a cylinder with radius r and height h that has one end removed is $(2\pi r^2 + 2\pi rh) - \pi r^2 = \pi r^2 + 2\pi rh$.
Therefore, the surface area of the given solid is

$$(\pi(8^2) + 2\pi(8)(7)) + (\pi(4^2) + 2\pi(4)(3)) + (\pi 8^2 - \pi 4^2) = 176\pi + 40\pi + 48\pi = 264\pi$$

(The term $\pi 8^2 - \pi 4^2$ represents the area of the top circular face of the larger cylinder minus the area of the top circular face of the smaller cylinder.)
Thus, the volume of the given solid is 400π and its surface area is 264π.

2. (a) The 1st ball goes in cup 1.
Each successive ball goes 5 cups further along the sequence. We obtain the cup number for the next ball by adding 5 to the current cup number. If the total is no larger than 12, this is the cup number of the next ball; if the cup number is larger than 12, we subtract 12 to obtain the actual cup number.
Thus, the 2nd ball goes in cup $1 + 5 = 6$, the 3rd ball goes in cup $6 + 5 = 11$, and the 4th ball goes in cup "$11 + 5 = 16$" which is $16 - 12 = 4$ cups beyond cup 12, so is really cup 4.
We continue until we place a second ball in cup 1.
In order, the balls are placed in cups 1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8, 1.

(b) The first ball goes in cup 1.
The next ball goes in cup $1 + 6 = 7$.
The next ball goes in cup $7 + 6 = 13$, which is the same as cup $13 - 9 = 4$.
The next ball goes in cup $4 + 6 = 10$, which is the same as cup $10 - 9 = 1$.
This means that the process stops.
Therefore, balls are placed in 3 cups in total (cups 1, 7, 4) and so cups 2, 3, 5, 6, 8, and 9 do not receive a ball.

(c) The first ball goes in cup 1.
Cup 1 will be reached again after moving 120 cups around the circle.
Since a ball is placed in every 3rd cup, then 120 cups corresponds to $\frac{120}{3} = 40$ additional balls, and the 41st ball will be placed in cup 1 and the process stops.
Therefore, there will be 40 cups in which a ball is placed. (Cup 1 will receive two balls.)
Since 40 cups receive a ball, then $120 - 40 = 80$ cups do not contain at least one ball.

(d) The first ball goes in cup 1.
Since each ball is placed 7 cups further along the circle, then the 338th ball will be placed $337 \times 7 = 2359$ cups further along the circle from cup 1.
Cup 1 itself is 1000 cups further along the circle, so we can think of moving 1000 cups to cup 1, then another 1000 cups to cup 1 again, and then 359 further cups to cup 360.
Therefore, the 338th ball is placed in cup 360.

3. (a) The differences between the given integers are

$$\begin{array}{cccc} 6-3=3 & 13-3=10 & 21-3=18 & 32-3=29 \\ & 13-6=7 & 21-6=15 & 32-6=26 \\ & & 21-13=8 & 32-13=19 \\ & & & 32-21=11 \end{array}$$

Therefore, the PDL is $3, 7, 8, 10, 11, 15, 18, 19, 26, 29$.

(b) Since $x > 16$, the differences between the given integers are

$$\begin{array}{cccc} 4-1=3 & 9-1=8 & 16-1=15 & x-1 \\ & 9-4=5 & 16-4=12 & x-4 \\ & & 16-9=7 & x-9 \\ & & & x-16 \end{array}$$

In terms of x, the sum of these differences is $50+(x-1)+(x-4)+(x-9)+(x-16)$ which equals $4x+20$.
Since these 10 integers also make up the PDL and we are told that the sum of the integers in the PDL is 112, then $4x+20=112$ and so $4x=92$ or $x=23$.

(c) Each of the four sets

$$\{3,5,10,11,14\} \qquad \{3,5,10,13,14\} \qquad \{3,6,7,12,14\} \qquad \{3,4,7,12,14\}$$

has a PDL that contains no repeated integers and so answers the given question.
Here is one method to find these sets.

We are given the set $\{3,q,r,s,14\}$ and the condition $3 < q < r < s < 14$.
We call the differences between adjacent numbers (eg. 3 and q, q and r, etc.) in the set "primary differences" and the differences between numbers with one number in between (3 and r, q and s, r and 14) "secondary differences".
Consider the four primary differences $q-3$, $r-q$, $s-r$, and $14-s$. Each of these is a positive integer. For the PDL to not contain any repeated integers, these primary differences must all be different.
Note also that $(q-3)+(r-q)+(s-r)+(14-s) = 14-3 = 11$. (The sum of the four primary differences equals the difference between the first and last numbers.)
The sum of four different positive integers is at least $1+2+3+4=10$. In order for the sum of four different positive integers to be 11, the integers must be 1, 2, 3, 5:

> If each of the integers was at least 2, their sum would be at least $2+3+4+5=14$, so one of the integers must be 1.
> If each of the three remaining integers was at least 3, their sum would be at least $1+3+4+5=13$, so one of the three remaining integers must be 2.
> If each of the two remaining integers was at least 4, their sum would be at least $1+2+4+5=12$, so one of the two remaining integers must be 3.
> Therefore, three of the integers are 1, 2, 3 so the fourth must be 5.

Therefore, the four primary differences must be 1, 2, 3, 5. We try to arrange these primary differences in such a way as to create a PDL with no repeated integers.
We note that each secondary difference is the sum of the two corresponding primary differences. (For example, $r-3=(q-3)+(r-q)$.)
Since we want all of the differences in the PDL to be different, then primary differences of 1 and 2 cannot be adjacent, otherwise we would create a secondary difference of $1+2=3$

which would duplicate the primary difference of 3.
Similarly, primary differences of 2 and 3 cannot be adjacent, otherwise we would create a secondary difference of $2 + 3 = 5$ which would duplicate the primary difference of 5.
Therefore, we must arrange the integers 1, 2, 3, 5 so that the 2 is not adjacent to either 1 or 3.
In this case, 2 must be the first or last primary difference (since it can only be next to one other primary difference) and must be adjacent to 5.
Suppose that 2 is the first primary difference. Then the ordered list of primary differences is either $2, 5, 1, 3$ or $2, 5, 3, 1$.
Starting with a first number 3 in the set, these primary differences would give either the set $\{3, 5, 10, 11, 14\}$ or the set $\{3, 5, 10, 13, 14\}$.
The first of these has PDL $1, 2, 3, 4, 5, 6, 7, 8, 9, 11$, all of which are different.
The second of these has PDL $1, 2, 3, 4, 5, 7, 8, 9, 10, 11$, all of which are different.
We can also reverse the order of the primary differences to obtain the sets $\{3, 6, 7, 12, 14\}$ and $\{3, 4, 7, 12, 14\}$ each of which has no repetition in its PDL.
Our argument above shows that these are the only possible sets.
Therefore, the four sets of the required form whose PDLs have no repetition are

$$\{3, 5, 10, 11, 14\} \qquad \{3, 5, 10, 13, 14\} \qquad \{3, 6, 7, 12, 14\} \qquad \{3, 4, 7, 12, 14\}$$

Any one of these sets answers the given question.

(d) Consider a set of the form $\{3, q, r, s, t\}$ with $3 < q < r < s < t$ and $t < 14$. We want to prove that the PDL of such a set always contains repeated integers.
We use the terms primary differences and secondary differences as in part (c).
If there is any repetition among the primary differences, then our result is true.
Therefore, we may suppose that there is no repetition among the primary differences.
Thus $q-3$, $r-q$, $s-r$, and $t-s$ are all different, and so their sum is at least $1+2+3+4 = 10$.
But $t - 3 = (q - 3) + (r - q) + (s - r) + (t - s)$ and this sum is at least 10.
Since $t < 14$, then $t - 3 < 11$. Therefore, $t - 3$ must equal 10, and so $t = 13$.
Further, this means that the primary differences must be 1, 2, 3, and 4 since the sum of the primary differences is 10.
Consider arranging these primary differences.
If 1 is adjacent to 2 or 3, we would form secondary differences of $2 + 1 = 3$ or $3 + 1 = 4$, which would mean that there is repetition in the PDL.
Suppose then that 1 is not adjacent to 2 or 3. This means that 1 must be only adjacent to 4, so must be at one end of the list of primary differences.
But this means that 2 and 3 are also adjacent. If 1 and 4 are adjacent and 2 and 3 are adjacent, then there are two primary differences of 5 (since $1 + 4 = 5$ and $2 + 3 = 5$).
Therefore, there is repetition in the PDL.
In other words, every such set produces repetition in the PDL.