The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Hypatia Contest

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

Solutions

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1. (a) In $\triangle PTQ$, $\angle PTQ = 90^\circ$.
Using the Pythagorean Theorem, $PQ^2 = 32^2 + 24^2$ or $PQ^2 = 1024 + 576 = 1600$ and so $PQ = \sqrt{1600} = 40$, since $PQ > 0$.

(b) In $\triangle QTR$, $\angle QTR = 90^\circ$.
Using the Pythagorean Theorem, $51^2 = TR^2 + 24^2$ or $TR^2 = 2601 - 576 = 2025$ and so $TR = \sqrt{2025} = 45$, since $TR > 0$.
Since $PT = 32$ and $TR = 45$, then $PR = PT + TR = 32 + 45 = 77$.
In $\triangle PQR$, QT is perpendicular to base PR and so $\triangle PQR$ has area $\frac{1}{2} \times (PR) \times (QT) = \frac{1}{2} \times 77 \times 24 = 924$.

(c) From part (b), the length of PR is 77.
Since $QS : PR = 12 : 11$, then $\dfrac{QS}{77} = \dfrac{12}{11}$ or $QS = 77 \times \dfrac{12}{11} = 84$.
So then $TS = QS - QT = 84 - 24 = 60$.
Using the Pythagorean Theorem in $\triangle PTS$, $PS = \sqrt{32^2 + 60^2}$ or $PS = \sqrt{4624} = 68$, since $PS > 0$.
Using the Pythagorean Theorem in $\triangle RTS$, $RS = \sqrt{45^2 + 60^2}$ or $RS = \sqrt{5625} = 75$, since $RS > 0$.
Thus, quadrilateral $PQRS$ has perimeter $40 + 51 + 75 + 68$ or 234.

2. (a) Expanding, $(a + b)^2 = a^2 + 2ab + b^2 = (a^2 + b^2) + 2ab$.
Since $a^2 + b^2 = 24$ and $ab = 6$, then $(a + b)^2 = 24 + 2(6) = 36$.

(b) Expanding, $(x + y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2xy$.
Since $(x + y)^2 = 13$ and $x^2 + y^2 = 7$, then $13 = 7 + 2xy$ or $2xy = 6$, and so $xy = 3$.

(c) Expanding, $(j + k)^2 = j^2 + 2jk + k^2 = (j^2 + k^2) + 2jk$.
Since $j + k = 6$ and $j^2 + k^2 = 52$, then $6^2 = 52 + 2jk$ or $2jk = -16$, and so $jk = -8$.

(d) Expanding, $(m^2 + n^2)^2 = m^4 + 2m^2n^2 + n^4 = (m^4 + n^4) + 2m^2n^2$.
Since $m^2 + n^2 = 12$ and $m^4 + n^4 = 136$, then $12^2 = 136 + 2m^2n^2$ or $2m^2n^2 = 8$ or $m^2n^2 = 4$, and so $mn = \pm 2$.

3. (a) Since $\angle MON = 90^\circ$, the product of the slopes of NO and OM is -1.
The slope of NO is $\dfrac{n^2 - 0}{n - 0} = n$, since $n \neq 0$ (points N and O are distinct).
The slope of OM is $\dfrac{\frac{1}{4} - 0}{\frac{1}{2} - 0} = \dfrac{1}{2}$.
Thus, $n \times \dfrac{1}{2} = -1$ or $n = -2$.

(b) Since $\angle ABO = 90^\circ$, the product of the slopes of BA and BO is -1.
The slope of BA is $\dfrac{b^2 - 4}{b - 2} = \dfrac{(b - 2)(b + 2)}{b - 2} = b + 2$, since $b \neq 2$ (A and B are distinct).
The slope of BO is $\dfrac{b^2 - 0}{b - 0} = b$, since $b \neq 0$ (B and O are distinct).
Thus, $(b + 2) \times b = -1$ or $b^2 + 2b + 1 = 0$.
Factoring, $(b + 1)(b + 1) = 0$ and so $b = -1$.

(c) Since $\angle PQR = 90^\circ$, the product of the slopes of PQ and RQ is -1.
The slope of PQ is $\dfrac{p^2 - q^2}{p - q} = \dfrac{(p - q)(p + q)}{p - q} = p + q$, since $p \neq q$ (P and Q are distinct).
The slope of RQ is $\dfrac{r^2 - q^2}{r - q} = \dfrac{(r - q)(r + q)}{r - q} = r + q$, since $r \neq q$ (R and Q are distinct).

Thus, $(p+q)\times(r+q)=-1$.
Since p,q and r are integers, then $p+q$ and $r+q$ are integers.
In order that $(p+q)\times(r+q)=-1$, either $p+q=1$ and $r+q=-1$ or $p+q=-1$ and $r+q=1$ (these are the only possibilities for integers p,q,r for which $(p+q)\times(r+q)=-1$).
In the first case, we add the two equations to get $p+q+r+q=1+(-1)$ or $2q+p+r=0$.
In the second case, adding the two equations gives $p+q+r+q=-1+1$ or $2q+p+r=0$.
In either case, $2q+p+r=0$, as required.

4. (a) Since p is an odd prime integer, then $p>2$.
Since the only prime divisors of $2p^2$ are 2 and p, then the positive divisors of $2p^2$ are $1,2,p,2p,p^2$, and $2p^2$.
So then, $S(2p^2)=1+2+p+2p+p^2+2p^2=3p^2+3p+3$.
Since $S(2p^2)=2613$, then $3p^2+3p+3=2613$ or $3p^2+3p-2610=0$ or $p^2+p-870=0$.
Factoring, $(p+30)(p-29)=0$, and so $p=29$ ($p\neq -30$ since p is an odd prime).

(b) Suppose $m=2p$ and $n=9q$ for some prime numbers $p,q>3$.
The positive divisors of $2p$, thus m, are $1,2,p$, and $2p$ (since $p>3$).
Therefore, $S(m)=1+2+p+2p=3p+3$.
The positive divisors of $9q$, thus n, are $1,3,q,3q,9$, and $9q$ (since $q>3$).
Therefore, $S(n)=1+3+9+q+3q+9q=13q+13$.
Since $S(m)=S(n)$, then $3p+3=13q+13$ or $3p-13q=10$.
Also, m and n are consecutive integers and so either $m-n=1$ or $n-m=1$.

If $m-n=1$, then $2p-9q=1$.
We solve the following system of two equations and two unknowns.

$$\begin{aligned} 2p-9q &= 1 \qquad (1)\\ 3p-13q &= 10 \qquad (2)\end{aligned}$$

Multiplying equation (1) by 3 and equation (2) by 2 we get,

$$\begin{aligned} 6p-27q &= 3 \qquad (3)\\ 6p-26q &= 20 \qquad (4)\end{aligned}$$

Subtracting equation (3) from equation (4), we get $q=17$.
Substituting $q=17$ into equation (1), $2p-9(17)=1$ or $2p=154$, and so $p=77$.
However, p must be a prime and thus $p\neq 77$.
There is no solution when $m-n=1$.

If $n-m=1$, then $9q-2p=1$.
We solve the following system of two equations and two unknowns.

$$\begin{aligned} 9q-2p &= 1 \qquad (5)\\ 3p-13q &= 10 \qquad (6)\end{aligned}$$

Multiplying equation (5) by 3 and equation (6) by 2 we get,

$$\begin{aligned} 27q-6p &= 3 \qquad (7)\\ 6p-26q &= 20 \qquad (8)\end{aligned}$$

Adding equation (7) and equation (8), we get $q=23$.
Substituting $q=23$ into equation (6), $3p-13(23)=10$ or $3p=309$, and so $p=103$.
Since $q=23$ and $p=103$ are prime integers greater than 3, then $m=2(103)=206$ and $n=9(23)=207$ are the only pair of consecutive integers satisfying the given properties.

(c) Since the only prime divisors of p^3q are p and q, then the positive divisors of p^3q, are $1, p, q, pq, p^2, p^2q, p^3$, and p^3q (since p and q are distinct primes).
Therefore, $S(p^3q) = p^3q + p^3 + p^2q + p^2 + pq + p + q + 1$.
Simplifying,

$$\begin{aligned} S(p^3q) &= p^3q + p^3 + p^2q + p^2 + pq + p + q + 1 \\ &= (p^3q + p^2q + pq + q) + (p^3 + p^2 + p + 1) \\ &= q(p^3 + p^2 + p + 1) + (p^3 + p^2 + p + 1) \\ &= (q + 1)(p^3 + p^2 + p + 1) \\ &= (q + 1)(p^2(p + 1) + (p + 1)) \\ &= (q + 1)(p + 1)(p^2 + 1) \end{aligned}$$

We are to determine the number of pairs of distinct primes p and q, each less than 30, such that $(q + 1)(p + 1)(p^2 + 1)$ is not divisible by 24.
There are 10 primes less than 30. These are $2, 3, 5, 7, 11, 13, 17, 19, 23$ and 29.
Therefore, the total number of possible pairs (p, q), where $p \neq q$, is $10 \times 9 = 90$.
We will count the number of pairs (p, q) for which $(q + 1)(p + 1)(p^2 + 1)$ *is* divisible by 24 and then subtract this total from 90.

If p or q equals 23, then 24 divides $(q + 1)(p + 1)(p^2 + 1)$.
There are 9 ordered pairs of the form $(23, q)$ and 9 of the form $(p, 23)$.
Thus, we count 18 pairs and since we have exhausted all possibilities using 23, we remove it from our list of 10 primes above.

Since $24 = 2^3 \times 3$, we can determine values of q for a given value of p by recognizing that each of these prime factors (three 2s and one 3) must occur in the prime factorization of $(q + 1)(p + 1)(p^2 + 1)$.
For example if $p = 2$, then $(q + 1)(p + 1)(p^2 + 1) = (q + 1)(3)(5)$.
Therefore, for $(q + 1)(p + 1)(p^2 + 1)$ to be a multiple of 24, $q + 1$ must be a multiple of 8 (since we are missing 2^3).
Thus when $p = 2$, the only possible value of q is 7 (we get this by trying the other 8 values in the list of primes).
We organize all possibilities for p (and the resulting values of q) in the table below.

p	$(p+1)(p^2+1)$	$q+1$ must be a multiple of	q (distinct from p)	Number of ordered pairs
2	$(3)(5)$	$2^3 = 8$	$q = 7$	1
3	$(4)(10) = 2^3 \times 5$	3	$q = 2, 5, 11, 17, 29$	5
5	$(6)(26) = 2^2 \times 3 \times 13$	2	$q = 3, 7, 11, 13, 17, 19, 29$	7
7	$(8)(50) = 2^3 \times 50$	3	$q = 2, 5, 11, 17, 29$	5
11	$(12)(122) = 2^3 \times 3 \times 61$	any q will work	$q = 2, 3, 5, 7, 13, 17, 19, 29$	8
13	$(14)(170) = 2^2 \times 595$	$2 \times 3 = 6$	$q = 5, 11, 17, 29$	4
17	$(18)(290) = 2^2 \times 3 \times 435$	2	$q = 3, 5, 7, 11, 13, 19, 29$	7
19	$(20)(362) = 2^3 \times 905$	3	$q = 2, 5, 11, 17, 29$	5
29	$(30)(842) = 2^2 \times 3 \times 2105$	2	$q = 3, 5, 7, 11, 13, 17, 19$	7

The total number of pairs (p, q) for which 24 divides $S(p^3q)$ is

$$18 + 1 + 5 + 7 + 5 + 8 + 4 + 7 + 5 + 7 = 67.$$

Thus, the total number of pairs of distinct prime integers p and q, each less than 30, such that $S(p^3q)$ *is not* divisible by 24, is $90 - 67 = 23$.