Hypatia Contest

(Grade 11)

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

UNIVERSITY OF
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Do not open this booklet until instructed to do so.

Time: 75 minutes

Number of questions: 4

Calculators are permitted

Each question is worth 10 marks

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 💡

   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided

2. **FULL SOLUTION** parts indicated by ✎

   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.
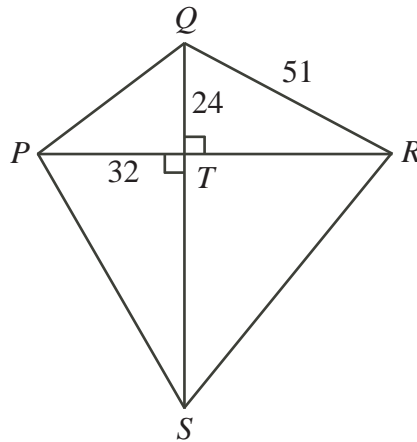
Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published in the FGH Results on our Web site, http://www.cemc.uwaterloo.ca.

1. Quadrilateral $PQRS$ is constructed with $QR = 51$, as shown. The diagonals of $PQRS$ intersect at $90°$ at point T, such that $PT = 32$ and $QT = 24$.

(a) Calculate the length of PQ.

(b) Calculate the area of $\triangle PQR$.

(c) If $QS : PR = 12 : 11$, determine the perimeter of quadrilateral $PQRS$.

2. (a) Determine the value of $(a + b)^2$, given that $a^2 + b^2 = 24$ and $ab = 6$.

(b) If $(x + y)^2 = 13$ and $x^2 + y^2 = 7$, determine the value of xy.

(c) If $j + k = 6$ and $j^2 + k^2 = 52$, determine the value of jk.

(d) If $m^2 + n^2 = 12$ and $m^4 + n^4 = 136$, determine all possible values of mn.

3. (a) Points $M(\frac{1}{2}, \frac{1}{4})$ and $N(n, n^2)$ lie on the parabola with equation $y = x^2$, as shown. Determine the value of n such that $\angle MON = 90°$.
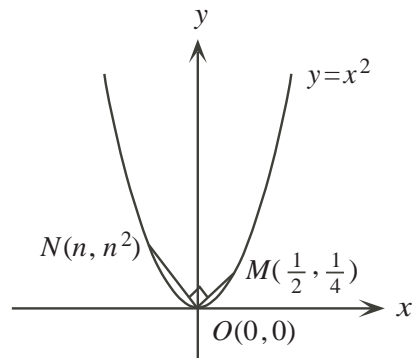
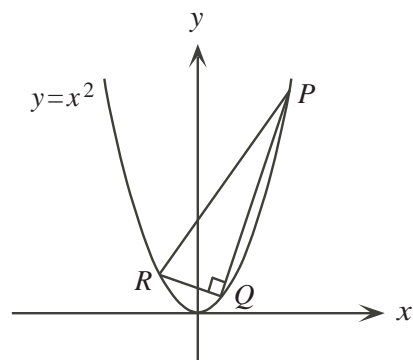(b) Points $A(2, 4)$ and $B(b, b^2)$ are the endpoints of a chord of the parabola with equation $y = x^2$, as shown. Determine the value of b so that $\angle ABO = 90°$.
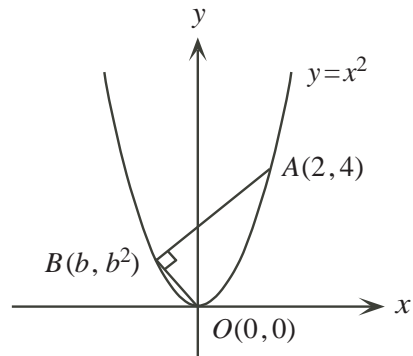
(c) Right-angled triangle PQR is inscribed in the parabola with equation $y = x^2$, as shown. Points P, Q and R have coordinates $(p, p^2), (q, q^2)$ and (r, r^2), respectively. If p, q and r are integers, show that $2q + p + r = 0$.

4. The positive divisors of 21 are 1, 3, 7 and 21. Let $S(n)$ be the sum of the positive divisors of the positive integer n. For example, $S(21) = 1 + 3 + 7 + 21 = 32$.

(a) If p is an odd prime integer, find the value of p such that $S(2p^2) = 2613$.

(b) The consecutive integers 14 and 15 have the property that $S(14) = S(15)$. Determine all pairs of consecutive integers m and n such that $m = 2p$ and $n = 9q$ for prime integers $p, q > 3$, and $S(m) = S(n)$.

(c) Determine the number of pairs of distinct prime integers p and q, each less than 30, with the property that $S(p^3q)$ is not divisible by 24.

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