The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Euclid Contest

Wednesday, April 11, 2012
(in North America and South America)

Thursday, April 12, 2012
(outside of North America and South America)

Solutions

1. (a) Since John buys 10 bags of apples, each of which contains 20 apples, then he buys a total of $10 \times 20 = 200$ apples.
Since he eats 8 apples a day, then it takes him $200 \div 8 = 25$ days to eat these apples.

(b) Evaluating,

$$\begin{aligned} & \sin(0^\circ) + \sin(60^\circ) + \sin(120^\circ) + \sin(180^\circ) + \sin(240^\circ) + \sin(300^\circ) + \sin(360^\circ) \\ = & \ 0 + \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{3}}{2} + 0 + (-\tfrac{\sqrt{3}}{2}) + (-\tfrac{\sqrt{3}}{2}) + 0 \\ = & \ 0 \end{aligned}$$

Alternatively, we could notice that $\sin(60^\circ) = -\sin(300^\circ)$ and $\sin(120^\circ) = -\sin(240^\circ)$ and $\sin(0^\circ) = \sin(180^\circ) = \sin(360^\circ) = 0$, so the sum is 0.

(c) Since the set of integers has a sum of 420 and an average of 60, then there are $420 \div 60 = 7$ integers in the set.
Since one integer is 120, then the remaining 6 integers have a sum of $420 - 120 = 300$ and so have an average of $300 \div 6 = 50$.

2. (a) Since $ax + ay = 4$, then $a(x + y) = 4$.
Since $x + y = 12$, then $12a = 4$ or $a = \frac{4}{12} = \frac{1}{3}$.

(b) Since the two lines are parallel, then their slopes are equal.
We re-write the given equations in the form "$y = mx + b$".
The first equation becomes $6y = -4x + 5$ or $y = -\frac{4}{6}x + \frac{5}{6}$ or $y = -\frac{2}{3}x + \frac{5}{6}$.
Since the first line is not vertical, then the second line is not vertical, and so $k \neq 0$.
The second equation becomes $ky = -6x + 3$ or $y = -\frac{6}{k}x + \frac{3}{k}$.
Therefore, $-\frac{2}{3} = -\frac{6}{k}$ and so $\frac{k}{6} = \frac{3}{2}$ or $k = 6 \times \frac{3}{2} = 9$.

(c) Adding the two equations, we obtain $x + x^2 = 2$ or $x^2 + x - 2 = 0$.
Factoring, we obtain $(x + 2)(x - 1) = 0$, and so $x = -2$ or $x = 1$.
From the first equation, $y = -x$. If $x = -2$, then $y = 2$ and if $x = 1$, then $y = -1$.
Therefore, the solutions are $(x, y) = (-2, 2)$ and $(x, y) = (1, -1)$.
(We can check that each of these solutions satisfies both equations.)

3. (a) Since the 200 g solution is 25% salt by mass, then $\frac{1}{4}$ of the mass (or 50 g) is salt and the rest (150 g) is water.
When water is added, the mass of salt does not change. Therefore, the 50 g of salt initially in the solution becomes 10% (or $\frac{1}{10}$) of the final solution by mass.
Therefore, the total mass of the final solution is $10 \times 50 = 500$ g.
Thus, the mass of water added is $500 - 200 = 300$ g.

(b) We are told that $F = \frac{9}{5}C + 32$.
From the given information $f = 2C + 30$.
We determine an expression for the error in terms of C by first determining when $f < F$.
The inequality $f < F$ is equivalent to $2C + 30 < \frac{9}{5}C + 32$ which is equivalent to $\frac{1}{5}C < 2$ which is equivalent to $C < 10$.
Therefore, $f < F$ precisely when $C < 10$.
Thus, for $-20 \leq C < 10$, the error equals $F - f = (\frac{9}{5}C + 32) - (2C + 30) = 2 - \frac{1}{5}C$.
Also, for $10 \leq C \leq 35$, the error equals $f - F = (2C + 30) - (\frac{9}{5}C + 32) = \frac{1}{5}C - 2$.
When $-20 \leq C < 10$, the error in terms of C is $2 - \frac{1}{5}C$ which is linear with negative slope, so is decreasing as C increases. Thus, the maximum value of error in this range for C occurs

when C is smallest, that is, when $C = -20$. This gives an error of $2 - \frac{1}{5}(-20) = 2 + 4 = 6$. When $10 \leq C \leq 35$, the error in terms of C is $\frac{1}{5}C - 2$ which is linear with positive slope, so is increasing as C increases. Thus, the maximum value of error in this range for C occurs when C is largest, that is, when $C = 35$. This gives an error of $\frac{1}{5}(35) - 2 = 7 - 2 = 5$.

Having considered the two possible ranges for C, the maximum possible error that Gordie would make is 6.

4. (a) *Solution 1*

Since the x-intercepts of the parabola with equation $y = 2(x-3)(x-5)$ are $x = 3$ and $x = 5$, then its axis of symmetry is at $x = \frac{1}{2}(3+5) = 4$.
If a horizontal line intersects the parabola at two points, then these points are symmetric across the axis of symmetry.
Since the line $y = k$ intersects the parabola at two points A and B with $AB = 6$, then each of A and B must be 3 units from the axis of symmetry.
Therefore, the x-coordinates of A and B are $4 - 3 = 1$ and $4 + 3 = 7$.
Thus, the coordinates of A and B, in some order, are $(1, k)$ and $(7, k)$.
Substituting $(1, k)$ into the equation of the parabola gives $k = 2(1-3)(1-5) = 16$.
(Substituting $(7, k)$ would give the same value of k.)

Solution 2

Let x_A be the x-coordinate of A and x_B be the x-coordinate of B. We may assume that A is to the left of B; that is, we assume that $x_A < x_B$. Since AB is horizontal and $AB = 6$, then $x_B - x_A = 6$.
Since A and B are the points of intersection between the line with equation $y = k$ and the parabola with equation $y = 2(x-3)(x-5)$, then we can solve for x_A and x_B by equating values of y to obtain the equation $k = 2(x-3)(x-5)$, which is equivalent to $k = 2(x^2 - 8x + 15)$ or $2x^2 - 16x + (30 - k) = 0$.
Using the quadratic formula, we obtain

$$x_A, x_B = \frac{16 \pm \sqrt{(-16)^2 - 4(2)(30-k)}}{2(2)}$$

Thus, $x_A = \dfrac{16 - \sqrt{16^2 - 4(2)(30-k)}}{2(2)}$ and $x_B = \dfrac{16 + \sqrt{16^2 - 4(2)(30-k)}}{2(2)}$.
Since $x_B - x_A = 6$, then

$$\begin{aligned}
\frac{16 + \sqrt{16^2 - 4(2)(30-k)}}{2(2)} - \frac{16 - \sqrt{16^2 - 4(2)(30-k)}}{2(2)} &= 6 \\
\frac{2\sqrt{16^2 - 4(2)(30-k)}}{2(2)} &= 6 \\
\sqrt{256 - (240 - 8k)} &= 12 \\
\sqrt{16 + 8k} &= 12 \\
16 + 8k &= 144 \\
8k &= 128 \\
k &= 16
\end{aligned}$$

Therefore, $k = 16$.
We can double check that the line with equation $y = 16$ intersects the parabola with equation $y = 2(x-3)(x-5)$ at the points $(1, 16)$ and $(7, 16)$, which are a distance 6 apart.

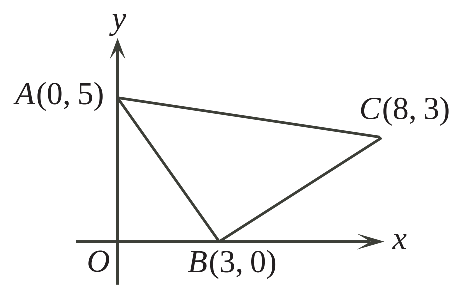
(b) Let $n = (3a + 6a + 9a + 12a + 15a) + (6b + 12b + 18b + 24b + 30b)$.
First, we simplify the given expression for n to obtain

$$n = (3a + 6a + 9a + 12a + 15a) + (6b + 12b + 18b + 24b + 30b) = 45a + 90b$$

We then factor the right side to obtain $n = 45(a + 2b) = 3^2 5^1 (a + 2b)$.
If $a + 2b = 5$, then $n = 3^2 5^2 = (3 \times 5)^2$, which is a perfect square.
Two pairs of positive integers (a, b) that satisfy $a + 2b = 5$ are $(a, b) = (3, 1)$ and $(a, b) = (1, 2)$.
Another value of $a + 2b$ for which n is a perfect square is $a + 2b = 20$, since here $n = 3^2 5^1 20 = 3^2 5^1 2^2 5^1 = 3^2 2^2 5^2 = (3 \times 2 \times 5)^2$.
A pair of positive integers (a, b) that satisfies $a + 2b = 20$ is $(18, 1)$.
Therefore, three pairs of positive integers (a, b) with the required property are $(3, 1), (1, 2), (18, 1)$.
(There are infinitely many other pairs with this property.)

5. (a) *Solution 1*
First, we calculate the side lengths of $\triangle ABC$:

$$\begin{aligned} AB &= \sqrt{(0-3)^2 + (5-0)^2} = \sqrt{34} \\ BC &= \sqrt{(3-8)^2 + (0-3)^2} = \sqrt{34} \\ AC &= \sqrt{(0-8)^2 + (5-3)^2} = \sqrt{68} \end{aligned}$$



Since $AB = BC$ and $AC = \sqrt{2}AB = \sqrt{2}BC$, then $\triangle ABC$ is an isosceles right-angled triangle, with the right angle at B.
Therefore, $\angle ACB = 45^\circ$.

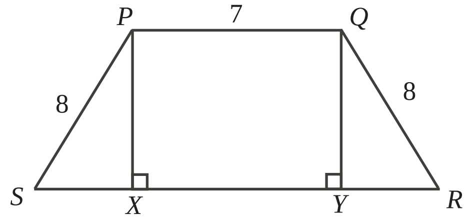
Solution 2
As in Solution 1, $AB = BC = \sqrt{34}$.
Line segment AB has slope $\frac{5-0}{0-3} = -\frac{5}{3}$.
Line segment BC has slope $\frac{0-3}{3-8} = \frac{3}{5}$.
Since the product of these two slopes is -1, then AB and BC are perpendicular.
Therefore, $\triangle ABC$ is right-angled at B.
Since $AB = BC$, then $\triangle ABC$ is an isosceles right-angled triangle, so $\angle ACB = 45^\circ$.

Solution 3
As in Solution 1, $AB = BC = \sqrt{34}$ and $AC = \sqrt{68}$.
Using the cosine law,

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2(AC)(BC)\cos(\angle ACB) \\ 34 &= 68 + 34 - 2(\sqrt{68})(\sqrt{34})\cos(\angle ACB) \\ 0 &= 68 - 2(\sqrt{2}\sqrt{34})(\sqrt{34})\cos(\angle ACB) \\ 0 &= 68 - 68\sqrt{2}\cos(\angle ACB) \\ 68\sqrt{2}\cos(\angle ACB) &= 68 \\ \cos(\angle ACB) &= \frac{1}{\sqrt{2}} \end{aligned}$$

Since $\cos(\angle ACB) = \frac{1}{\sqrt{2}}$ and $0^\circ < \angle ACB < 180^\circ$, then $\angle ACB = 45^\circ$.

(b) Draw perpendiculars from P and Q to X and Y, respectively, on SR.



Since PQ is parallel to SR (because $PQRS$ is a trapezoid) and PX and QY are perpendicular to SR, then $PQYX$ is a rectangle.
Thus, $XY = PQ = 7$ and $PX = QY$.
Since $\triangle PXS$ and $\triangle QYR$ are right-angled with $PS = QR$ and $PX = QY$, then these triangles are congruent, and so $SX = YR$.
Since $XY = 7$ and $SR = 15$, then $SX + 7 + YR = 15$ or $2 \times SX = 8$ and so $SX = 4$.
By the Pythagorean Theorem in $\triangle PXS$,

$$PX^2 = PS^2 - SX^2 = 8^2 - 4^2 = 64 - 16 = 48$$

Now PR is the hypotenuse of right-angled $\triangle PXR$.
Since $PR > 0$, then by the Pythagorean Theorem,

$$PR = \sqrt{PX^2 + XR^2} = \sqrt{48 + (7+4)^2} = \sqrt{48 + 11^2} = \sqrt{48 + 121} = \sqrt{169} = 13$$

Therefore, $PR = 13$.

6. (a) *Solution 1*
There are two possibilities: either each player wins three games or one player wins more games than the other.
Since the probability that each player wins three games is $\frac{5}{16}$, then the probability that any one player wins more games than the other is $1 - \frac{5}{16} = \frac{11}{16}$.
Since each of Blaise and Pierre is equally likely to win any given game, then each must be equally likely to win more games than the other.
Therefore, the probability that Blaise wins more games than Pierre is $\frac{1}{2} \times \frac{11}{16} = \frac{11}{32}$.

Solution 2
We consider the results of the 6 games as a sequence of 6 Bs or Ps, with each letter a B if Blaise wins the corresponding game or P if Pierre wins.
Since the two players are equally skilled, then the probability that each wins a given game is $\frac{1}{2}$. This means that the probability of each letter being a B is $\frac{1}{2}$ and the probability of each letter being a P is also $\frac{1}{2}$.
Since each sequence consists of 6 letters, then the probability of a particular sequence occurring is $(\frac{1}{2})^6 = \frac{1}{64}$, because each of the letters is specified.
Since they play 6 games in total, then the probability that Blaise wins more games than Pierre is the sum of the probabilities that Blaise wins 4 games, that Blaise wins 5 games, and that Blaise wins 6 games.
If Blaise wins 6 games, then the sequence consists of 6 Bs. The probability of this is $\frac{1}{64}$, since there is only one way to arrange 6 Bs.
If Blaise wins 5 games, then the sequence consists of 5 Bs and 1 P. The probability of this is $6 \times \frac{1}{64} = \frac{6}{64}$, since there are 6 possible positions in the list for the 1 P (eg. PBBBBB,

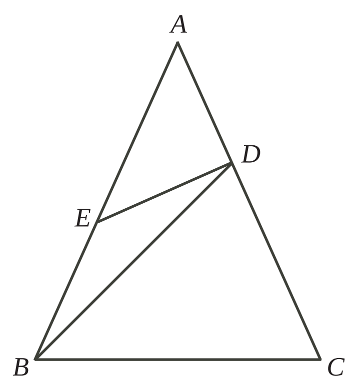
BPBBBB, BBPBBB, BBBPBB, BBBBPB, BBBBBP).
The probability that Blaise wins 4 games is $\binom{6}{2} \times \frac{1}{64} = \frac{15}{64}$, since there are $\binom{6}{2} = 15$ ways for 4 Bs and 2 Ps to be arranged.
Therefore, the probability that Blaise wins more games than Pierre is $\frac{1}{64}+\frac{6}{64}+\frac{15}{64} = \frac{22}{64} = \frac{11}{32}$.

(b) Using exponent rules and arithmetic, we manipulate the given equation:

$$\begin{aligned} 3^{x+2} + 2^{x+2} + 2^x &= 2^{x+5} + 3^x \\ 3^x 3^2 + 2^x 2^2 + 2^x &= 2^x 2^5 + 3^x \\ 9(3^x) + 4(2^x) + 2^x &= 32(2^x) + 3^x \\ 8(3^x) &= 27(2^x) \\ \frac{3^x}{2^x} &= \frac{27}{8} \\ \left(\frac{3}{2}\right)^x &= \left(\frac{3}{2}\right)^3 \end{aligned}$$

Since the two expressions are equal and the bases are equal, then the exponents must be equal, so $x = 3$.

7. (a) Since $AB = AC$, then $\triangle ABC$ is isosceles and $\angle ABC = \angle ACB$. Note that $\angle BAC = \theta$.

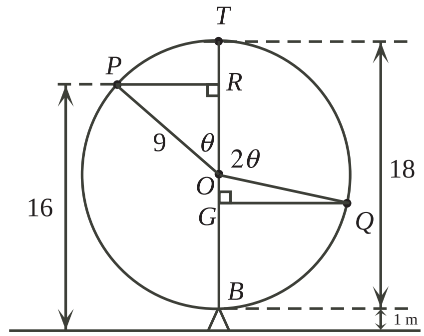


The angles in $\triangle ABC$ add to 180°, so $\angle ABC + \angle ACB + \angle BAC = 180^\circ$.
Thus, $2\angle ACB + \theta = 180^\circ$ or $\angle ABC = \angle ACB = \frac{1}{2}(180^\circ - \theta) = 90^\circ - \frac{1}{2}\theta$.
Now $\triangle BCD$ is isosceles as well with $BC = BD$ and so $\angle CDB = \angle DCB = 90^\circ - \frac{1}{2}\theta$.
Since the angles in $\triangle BCD$ add to 180°, then

$$\angle CBD = 180^\circ - \angle DCB - \angle CDB = 180^\circ - (90^\circ - \tfrac{1}{2}\theta) - (90^\circ - \tfrac{1}{2}\theta) = \theta$$

Now $\angle EBD = \angle ABC - \angle DBC = (90^\circ - \frac{1}{2}\theta) - \theta = 90^\circ - \frac{3}{2}\theta$.
Since $BE = ED$, then $\angle EDB = \angle EBD = 90^\circ - \frac{3}{2}\theta$.
Therefore, $\angle BED = 180^\circ - \angle EBD - \angle EDB = 180^\circ - (90^\circ - \frac{3}{2}\theta) - (90^\circ - \frac{3}{2}\theta) = 3\theta$.

(b) Let O be the centre of the ferris wheel and B the lowest point on the wheel.
Since the radius of the ferris wheel is 9 m (half of the diameter of 18 m) and B is 1 m above the ground, then O is $9 + 1 = 10$ m above the ground.
Let $\angle TOP = \theta$.

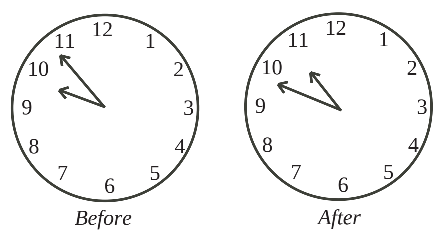


Since the ferris wheel rotates at a constant speed, then in 8 seconds, the angle through which the wheel rotates is twice the angle through which it rotates in 4 seconds. In other words, $\angle TOQ = 2\theta$.
Draw a perpendicular from P to R on TB and from Q to G on TB.
Since P is 16 m above the ground and O is 10 m above the ground, then $OR = 6$ m.
Since OP is a radius of the circle, then $OP = 9$ m.
Looking at right-angled $\triangle ORP$, we see that $\cos\theta = \dfrac{OR}{OP} = \dfrac{6}{9} = \dfrac{2}{3}$.
Since $\cos\theta = \frac{2}{3} < \frac{1}{\sqrt{2}} = \cos(45^\circ)$, then $\theta > 45^\circ$.
This means that $2\theta > 90^\circ$, which means that Q is below the horizontal diameter through O and so G is below O.
Since $\angle TOQ = 2\theta$, then $\angle QOG = 180^\circ - 2\theta$.
Kolapo's height above the ground at Q equals 1 m plus the length of BG.
Now $BG = OB - OG$. We know that $OB = 9$ m.
Also, considering right-angled $\triangle QOG$, we have

$$OG = OQ\cos(\angle QOG) = 9\cos(180^\circ - 2\theta) = -9\cos(2\theta) = -9(2\cos^2\theta - 1)$$

Since $\cos\theta = \frac{2}{3}$, then $OG = -9(2(\frac{2}{3})^2 - 1) = -9(\frac{8}{9} - 1) = 1$ m.
Therefore, $BG = 9 - 1 = 8$ m and so Q is $1 + 8 = 9$ m above the ground.

8. (a) *Solution 1*
The hour hand and minute hand both turn at constant rates. Since the hour hand moves $\frac{1}{12}$ of the way around the clock in 1 hour and the minute hand moves all of the way around the clock in 1 hour, then the minute hand turns 12 times as quickly as the hour hand.



Suppose also that the hour hand moves through an angle of x° between Before and After. Therefore, the minute hand moves through an angle of $(360^\circ - x^\circ)$ between Before and After, since these two angles add to 360°.

Since the minute hand moves 12 times as quickly as the hour hand, then $\dfrac{360^\circ - x^\circ}{x^\circ} = 12$ or $360 - x = 12x$ and so $13x = 360$, or $x = \frac{360}{13}$.
In one hour, the hour hand moves through $\frac{1}{12} \times 360^\circ = 30^\circ$.
Since the hour hand is moving for t hours, then we have $30^\circ t = (\frac{360}{13})^\circ$ and so $t = \frac{360}{30(13)} = \frac{12}{13}$.

Solution 2
Suppose that Jimmy starts painting x hours after 9:00 a.m. and finishes painting y hours after 10:00 a.m., where $0 < x < 1$ and $0 < y < 1$.
Since t is the amount of time in hours that he spends painting, then $t = (1-x)+y$, because he paints for $(1 - x)$ hours until 10:00 a.m., and then for y hours until his finishing time.
The hour hand and minute hand both turn at constant rates.
The minute hand turns 360° in one hour and the hour hand turns $\frac{1}{12} \times 360^\circ = 30^\circ$ in one hour.
Thus, in x hours, where $0 < x < 1$, the minute hand turns $(360x)^\circ$ and the hour hand turns $(30x)^\circ$.
In the Before picture, the minute hand is $(360x)^\circ$ clockwise from the 12 o'clock position.
In the After picture, the minute hand is $(360y)^\circ$ clockwise from the 12 o'clock position.
The 9 is $9 \times 30^\circ = 270^\circ$ clockwise from the 12 o'clock position and the 10 is $10 \times 30^\circ = 300^\circ$ clockwise from the 12 o'clock position.
Therefore, in the Before picture, the hour hand is $270^\circ + (30x)^\circ$ clockwise from the 12 o'clock position, and in the After picture, the hour hand is $300^\circ + (30y)^\circ$ clockwise from the 12 o'clock position.
Because the hour and minute hands have switched places from the Before to the After positions, then we can equate the corresponding positions to obtain $(360x)^\circ = 300^\circ + (30y)^\circ$ (or $360x = 300 + 30y$) and $(360y)^\circ = 270^\circ + (30x)^\circ$ (or $360y = 270 + 30x$).
Dividing both equations by 30, we obtain $12x = 10 + y$ and $12y = 9 + x$.
Subtracting the second equation from the first, we obtain $12x - 12y = 10 + y - 9 - x$ or $-1 = 13y - 13x$.
Therefore, $y - x = -\frac{1}{13}$ and so $t = (1 - x) + y = 1 + y - x = 1 - \frac{1}{13} = \frac{12}{13}$.

(b) We manipulate the given equation into a sequence of equivalent equations:

$$\begin{aligned}
\log_{5x+9}(x^2 + 6x + 9) + \log_{x+3}(5x^2 + 24x + 27) &= 4 \\
\frac{\log(x^2 + 6x + 9)}{\log(5x + 9)} + \frac{\log(5x^2 + 24x + 27)}{\log(x + 3)} &= 4 \qquad \text{(using the “change of base” formula)} \\
\frac{\log((x + 3)^2)}{\log(5x + 9)} + \frac{\log((5x + 9)(x + 3))}{\log(x + 3)} &= 4 \qquad \text{(factoring)} \\
\frac{2\log(x + 3)}{\log(5x + 9)} + \frac{\log(5x + 9) + \log(x + 3)}{\log(x + 3)} &= 4 \qquad \text{(using logarithm rules)} \\
2\left(\frac{\log(x + 3)}{\log(5x + 9)}\right) + \frac{\log(5x + 9)}{\log(x + 3)} + \frac{\log(x + 3)}{\log(x + 3)} &= 4 \qquad \text{(rearranging fractions)}
\end{aligned}$$

Making the substitution $t = \dfrac{\log(x+3)}{\log(5x+9)}$, we obtain successively

$$\begin{aligned} 2t + \frac{1}{t} + 1 &= 4 \\ 2t^2 + 1 + t &= 4t \\ 2t^2 - 3t + 1 &= 0 \\ (2t-1)(t-1) &= 0 \end{aligned}$$

Therefore, $t = 1$ or $t = \frac{1}{2}$.

If $\dfrac{\log(x+3)}{\log(5x+9)} = 1$, then $\log(x+3) = \log(5x+9)$ or $x + 3 = 5x + 9$, which gives $4x = -6$ or $x = -\frac{3}{2}$.

If $\dfrac{\log(x+3)}{\log(5x+9)} = \dfrac{1}{2}$, then $2\log(x+3) = \log(5x+9)$ or $\log((x+3)^2) = \log(5x+9)$ or $(x+3)^2 = 5x+9$.

Here, $x^2 + 6x + 9 = 5x + 9$ or $x^2 + x = 0$ or $x(x+1) = 0$, and so $x = 0$ or $x = -1$.

Therefore, there are three possible values for x: $x = 0$, $x = -1$ and $x = -\frac{3}{2}$.

We should check each of these in the original equation.

If $x = 0$, the left side of the original equation is $\log_9 9 + \log_3 27 = 1 + 3 = 4$.

If $x = -1$, the left side of the original equation is $\log_4 4 + \log_2 8 = 1 + 3 = 4$.

If $x = -\frac{3}{2}$, the left side of the original equation is $\log_{3/2}(9/4) + \log_{3/2}(9/4) = 2 + 2 = 4$.

Therefore, the solutions are $x = 0, -1, -\frac{3}{2}$.

9. (a) Suppose that the auditorium with these properties has r rows and c columns of chairs.
Then there are rc chairs in total.
Each chair is empty, is occupied by a boy, or is occupied by a girl.
Since there are 14 boys in each row, then there are $14r$ chairs occupied by boys.
Since there are 10 girls in each column, then there are $10c$ chairs occupied by girls.
Since there are exactly 3 empty chairs, then the total number of chairs can also be written as $14r + 10c + 3$.
Therefore, $rc = 14r + 10c + 3$.
We proceed to find all pairs of positive integers r and c that satisfy this equation. We note that since there are 14 boys in each row, then there must be at least 14 columns (that is, $c \geq 14$) and since there are 10 girls in each column, then there must be at least 10 rows (that is, $r \geq 10$).
Manipulating the equation,

$$\begin{aligned} rc &= 14r + 10c + 3 \\ rc - 14r &= 10c + 3 \\ r(c-14) &= 10c + 3 \\ r &= \frac{10c+3}{c-14} \\ r &= \frac{10c - 140 + 143}{c-14} \\ r &= \frac{10c-140}{c-14} + \frac{143}{c-14} \\ r &= 10 + \frac{143}{c-14} \end{aligned}$$

Since r is an integer, then $10 + \frac{143}{c-14}$ is an integer, so $\frac{143}{c-14}$ must be an integer.
Therefore, $c - 14$ is a divisor of 143. Since $c \geq 14$, then $c - 14 \geq 0$, so $c - 14$ is a positive divisor of 143.
Since $143 = 11 \times 13$, then its positive divisors are $1, 11, 13, 143$.
We make a table of the possible values of $c - 14$ along with the resulting values of c, r (calculated using $r = 10 + \frac{143}{c-14}$) and rc:

$c-14$	c	r	rc
1	15	153	2295
11	25	23	575
13	27	21	567
143	157	11	1727

Therefore, the four possible values for rc are $567, 575, 1727, 2295$. That is, the smallest possible number of chairs in the auditorium is 567.
(Can you create a grid with 27 columns and 21 rows that has the required properties?)

(b) *Solution 1*

We use the notation $|PMQN|$ to represent the area of quadrilateral $|PMQN|$, $|\triangle APD|$ to represent the area of $\triangle APD$, and so on.
We want to show that $|PMQN| = |\triangle APD| + |\triangle BQC|$.
This is equivalent to showing

$$|PMQN| + |\triangle DPN| + |\triangle CQN| = |\triangle APD| + |\triangle DPN| + |\triangle BQC| + |\triangle CQN|$$

which is equivalent to showing

$$|\triangle DMC| = |\triangle DAN| + |\triangle CBN|$$

since combining quadrilateral $PMQN$ with $\triangle DPN$ and $\triangle CQN$ gives $\triangle DMC$, combining $\triangle APD$ with $\triangle DPN$ gives $\triangle DAN$, and combining $\triangle BQC$ with $\triangle CQN$ gives $\triangle CBN$.
Suppose that DC has length x and DN has length tx for some t with $0 < t < 1$.
Then $NC = DC - DN = x - tx = (1-t)x$.
Suppose also that the height of A above DC is a, the height of B above DC is b and the height of M above DC is m.

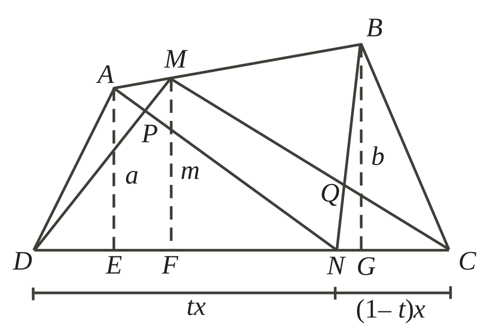


Figure 1

Then $|\triangle DAN| = \frac{1}{2}(tx)(a)$ and $|\triangle CBN| = \frac{1}{2}((1-t)x)b$ so

$$|\triangle DAN| + |\triangle CBN| = \tfrac{1}{2}(txa + (1-t)xb) = \tfrac{1}{2}x(ta + (1-t)b)$$

Also, $|\triangle DMC| = \frac{1}{2}xm$.
In order to prove that $|\triangle DMC| = |\triangle DAN| + |\triangle CBN|$, we need to show that $\frac{1}{2}xm$ equals

$\frac{1}{2}x(ta+(1-t)b)$ which is equivalent to showing that m is equal to $ta+(1-t)b$.

In Figure 2, we draw a horizontal line from A to BG, meeting MF at R and BG at S.

Since MF and BG are vertical and ARS is horizontal, then these line segments are perpendicular.

Since $AE=a$, $MF=m$ and $BG=b$, then $MR=m-a$ and $BS=b-a$.

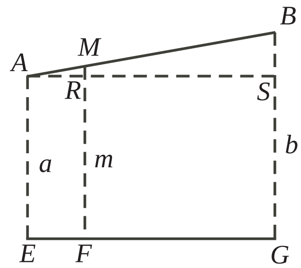


Figure 2

Now $\triangle ARM$ is similar to $\triangle ASB$, since each is right-angled and they share a common angle at A.

Therefore, $\dfrac{MR}{BS}=\dfrac{AM}{AB}=\dfrac{NC}{DC}$.

Since $MR=m-a$ and $BS=b-a$, then $\dfrac{MR}{BS}=\dfrac{m-a}{b-a}$.

Since $\dfrac{AM}{AB}=\dfrac{NC}{DC}$, then $\dfrac{MR}{BS}=\dfrac{(1-t)x}{x}=1-t$.

Comparing these two expressions, we obtain $\dfrac{m-a}{b-a}=(1-t)$ or $m-a=(b-a)(1-t)$ or $m=a+b(1-t)+(t-1)a=ta+(1-t)b$, as required.

This concludes the proof, and so $|PMQN|=|\triangle APD|+|\triangle BQC|$, as required.

Solution 2

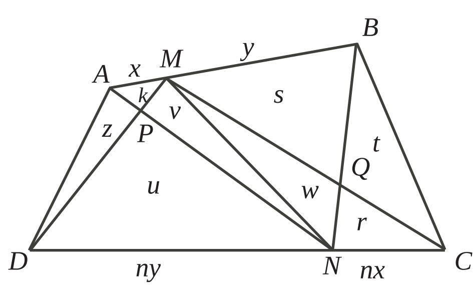
Let $AM=x$ and $MB=y$. Then $AB=x+y$ and so $\dfrac{AM}{AB}=\dfrac{x}{x+y}$.

Let $NC=nx$ for some real number n.

Since $\dfrac{NC}{DC}=\dfrac{AM}{AB}$, then $\dfrac{nx}{DC}=\dfrac{x}{x+y}$ and so $DC=n(x+y)$.

This tells us that $DN=DC-NC=n(x+y)-nx=ny$.

Join M to N and label the areas as shown in the diagram:

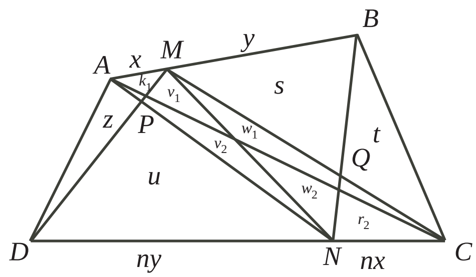


We repeatedly use the fact that triangles with a common height have areas in proportion to the lengths of their bases.

For example, $\triangle MDN$ and $\triangle MNC$ have a common height from line segment to DC to M and so the ratio of their areas equals the ratio of the lengths of their bases.

In other words, $\dfrac{w+r}{u+v}=\dfrac{nx}{ny}=\dfrac{x}{y}$. Thus, $w+r=\dfrac{x}{y}(u+v)$.

Also, the ratio of the area of $\triangle NAM$ to the area of $\triangle NMB$ equals the ratio of AM to MB.
This gives $\dfrac{k+v}{s+w} = \dfrac{x}{y}$ or $k+v = \dfrac{x}{y}(s+w)$.
Next, we join A to C and relabel the areas divided by this new line segment as shown:



(The unlabelled triangle adjacent to the one labelled k_1 has area k_2 and the unlabelled triangle adjacent to the one labelled r_2 has area r_1.)
Consider $\triangle ANC$ and $\triangle ADN$.
As above, the ratio of their areas equals the ratio of their bases.
Thus, $\dfrac{k_2+v_2+w_2+r_2}{z+u} = \dfrac{nx}{ny} = \dfrac{x}{y}$, and so $k_2+v_2+w_2+r_2 = \dfrac{x}{y}(z+u)$.
Consider $\triangle CAM$ and $\triangle CMB$.
As above, the ratio of their areas equals the ratio of their bases.
Thus, $\dfrac{k_1+v_1+w_1+r_1}{s+t} = \dfrac{x}{y}$, and so $k_1+v_1+w_1+r_1 = \dfrac{x}{y}(s+t)$.
Adding $k_2+v_2+w_2+r_2 = \dfrac{x}{y}(z+u)$ and $k_1+v_1+w_1+r_1 = \dfrac{x}{y}(s+t)$ gives

$$(k_1+k_2)+(v_1+v_2)+(w_1+w_2)+(r_1+r_2) = \frac{x}{y}(s+t+z+u)$$

or

$$k+v+w+r = \frac{x}{y}(s+t+z+u)$$

Since $k+v = \dfrac{x}{y}(s+w)$ and $w+r = \dfrac{x}{y}(u+v)$, then

$$\frac{x}{y}(s+w) + \frac{x}{y}(u+v) = \frac{x}{y}(s+t+z+u)$$

which gives

$$s+w+u+v = s+t+z+u$$

or

$$w+v = t+z$$

But $w+v$ is the area of quadrilateral $PMQN$, z is the area of $\triangle APD$ and t is the area of $\triangle BQC$. In other words, the area of quadrilateral $PMQN$ equals the sum of the areas of $\triangle APD$ and $\triangle PQC$, as required.

10. (a) The Eden sequences from $\{1, 2, 3, 4, 5\}$ are

$$1 \quad 3 \quad 5 \quad 1,2 \quad 1,4 \quad 3,4 \quad 1,2,3 \quad 1,2,5 \quad 1,4,5 \quad 3,4,5 \quad 1,2,3,4 \quad 1,2,3,4,5$$

There are 12 such sequences.

We present a brief justification of why these are all of the sequences.

* An Eden sequence of length 1 consists of a single odd integer. The possible choices are 1 and 3 and 5.
* An Eden sequence of length 2 consists of an odd integer followed by a larger even integer. Since the only possible even integers here are 2 and 4, then the possible sequences are 1, 2 and 1, 4 and 3, 4.
* An Eden sequence of length 3 starts with an Eden sequence of length 2 and appends (that is, adds to the end) a larger odd integer. Starting with 1,2, we form 1,2,3 and 1,2,5. Starting with 1,4, we form 1,4,5. Starting with 3,4, we form 3,4,5.
* An Eden sequence of length 4 starts with an Eden sequence of length 3 and appends a larger even integer. Since 2 and 4 are the only possible even integers, then the only possible sequence here is 1,2,3,4.
* An Eden sequence of length 5 from $\{1, 2, 3, 4, 5\}$ must include all 5 elements, so is 1,2,3,4,5.

(b) We will prove that, for all positive integers $n \geq 3$, we have $e(n) = e(n-1) + e(n-2) + 1$. Thus, if $e(18) = m$, then $e(19) = e(18) + e(17) + 1 = m + 4181$ and

$$e(20) = e(19) + e(18) + 1 = (m + 4181) + m + 1$$

Since $e(20) = 17710$, then $17710 = 2m + 4182$ or $2m = 13528$ and so $m = 6764$.
Therefore, $e(18) = 6764$ and $e(19) = 6764 + 4181 = 10945$.

So we must prove that, for all positive integers $n \geq 3$, we have $e(n) = e(n-1)+e(n-2)+1$.

To simplify the reading, we use a number of abbreviations:

* ES means "Eden sequence"
* ES(m) means "Eden sequence from $\{1, 2, 3, \dots, m\}$
* ESE and ESO mean "Eden sequence of even length" and "Eden sequence of odd length", respectively
* ESE(m) and ESO(m) mean "Eden sequence of even length from $\{1, 2, 3, \dots, m\}$" and "Eden sequence of odd length from $\{1, 2, 3, \dots, m\}$", respectively

<u>Method 1</u>
For each positive integer n, let $A(n)$ be the number of ESE(n), and let $B(n)$ be the number of ESO(n).
Then $e(n) = A(n) + B(n)$ for each positive integer n.
Note also that for each positive integer $n \geq 2$, we have $e(n) \geq e(n-1)$ and $A(n) \geq A(n-1)$ and $B(n) \geq B(n-1)$. This is because every ES($n-1$) is also an ES(n) because it satisfies the three required conditions. So there are at least as many ES(n) as there are ES($n-1$). (The same argument works to show that there are at least as many ESE(n) as there are ESE($n-1$), and at least as many ESO(n) as there are ESO($n-1$).)

Note that if k is a positive integer, then $2k + 1$ is odd and $2k$ is even.
The following four facts are true for every positive integer $k \geq 1$:

(i) $A(2k+1) = A(2k)$

(ii) $B(2k) = B(2k-1)$

(iii) $A(2k) = A(2k-1) + B(2k-1)$

(iv) $B(2k+1) = A(2k) + B(2k) + 1$

Here are justifications for these facts:

(i) An ESE must end with an even integer. Thus, an ESE$(2k+1)$ cannot include $2k+1$, since it would then have to include a larger even positive integer, which it cannot.
Therefore, an ESE$(2k+1)$ has largest term at most $2k$ and so is an ES$(2k)$.
Thus, $A(2k+1) \leq A(2k)$.
But from above, $A(2k+1) \geq A(2k)$, and so $A(2k+1) = A(2k)$.

(ii) An ESO must end with an odd integer. Thus, an ESO$(2k)$ cannot include $2k$, since it would then have to include a larger odd positive integer, which it cannot.
Therefore, an ESO$(2k)$ has largest term at most $2k-1$ and so is an ESO$(2k-1)$.
Thus, $B(2k) \leq B(2k-1)$.
But from above, $B(2k) \geq B(2k-1)$, and so $B(2k) = B(2k-1)$.

(iii) An ESE$(2k)$ either includes $2k$ or does not include $2k$.
If such a sequence includes $2k$, then removing the $2k$ produces an ESO$(2k-1)$. Also, every ESO$(2k-1)$ can be produced in this way.
Therefore, the number of sequences in this case is $B(2k-1)$.
If such a sequence does not include $2k$, then the sequence can be thought of as an ESE$(2k-1)$. Note that every ESE$(2k-1)$ is an ESE$(2k)$.
Therefore, the number of sequences in this case is $A(2k-1)$.
Thus, $A(2k) = A(2k-1) + B(2k-1)$.

(iv) An ESO$(2k+1)$ is either the one term sequence $2k+1$, or includes $2k+1$ and more terms, or does not include $2k+1$.
There is 1 sequence of the first kind.
As in (iii), there are $A(2k)$ sequences of the second kind and $B(2k)$ sequences of the third kind.
Thus, $B(2k+1) = 1 + A(2k) + B(2k)$.

Combining these facts, for each positive integer k, we obtain

$$\begin{aligned}
e(2k+1) &= A(2k+1) + B(2k+1) \\
&= A(2k) + (A(2k) + B(2k) + 1) \\
&= (A(2k) + B(2k)) + A(2k) + 1 \\
&= e(2k) + (A(2k-1) + B(2k-1)) + 1 \\
&= e(2k) + e(2k-1) + 1
\end{aligned}$$

and

$$\begin{aligned}
e(2k) &= A(2k) + B(2k) \\
&= (A(2k-1) + B(2k-1)) + B(2k-1) \\
&= e(2k-1) + (A(2k-2) + B(2k-2) + 1) \\
&= e(2k-1) + e(2k-2) + 1
\end{aligned}$$

Therefore, for all positive integers $n \geq 3$, we have $e(n) = e(n-1) + e(n-2) + 1$, as required.

Method 2

Let n be a positive integer with $n \geq 3$, and consider the ES(n).
We divide the sequences into three sets:

(i) The sequence 1 (there is 1 such sequence)

(ii) The sequences which begin with 1 and have more than 1 term

(iii) The sequences which do not begin with 1

We show that in case (ii) there are $e(n-1)$ sequences and in case (iii) there are $e(n-2)$ sequences. This will show that $e(n) = 1 + e(n-1) + e(n-2)$, as required.

(ii) Consider the set of ES(n) that begin with 1. We call this set of sequences P.
We remove the 1 from each of these and consider the set of resulting sequences. We call this set Q. Note that the number of sequences in P and in Q is the same.
Each of the sequences in Q includes numbers from the set $\{2, 3, \ldots, n\}$, is increasing, and has even terms in odd positions and odd terms in even positions (since each term has been shifted one position to the left).
The sequences in Q are in a one-to-one correspondence with the ES($n-1$) (we call this set of sequences R) and so there are exactly $e(n-1)$ of them (and so $e(n-1)$ sequences in P).
We can show that this one-to-one correspondence exists by subtracting 1 from each term of each sequence in Q, to form a set of sequences S. Each of the resulting sequences is distinct, includes numbers from the set $\{1, 2, 3, \ldots, n-1\}$, is increasing, and has odd terms in odd positions and even terms in even positions (since each term has been reduced by 1). Also, each sequence in R can be obtained in this way (since adding 1 to each term in one of these ES gives a distinct sequence in Q).
Therefore, the number of sequences in this case is $e(n-1)$.

(iii) Consider the set of ES(n) that do not begin with 1. We call this set of sequences T.
Since each sequence in T does not begin with 1, then the minimum number in each sequence is 3.
Thus, each of the sequences in T includes numbers from the set $\{3, 4, \ldots, n\}$, is increasing, and has odd terms in odd positions and even terms in even positions.
The sequences in T are in a one-to-one correspondence with the ES($n-2$) (we call this set of sequences U) and so there are exactly $e(n-2)$ of them.
We can show that this one-to-one correspondence exists by subtracting 2 from each term of each sequence in T, to form a set of sequences V. Each of the resulting sequences is distinct, includes numbers from the set $\{1, 2, 3, \ldots, n-2\}$, is increasing, and has odd terms in odd positions and even terms in even positions (since each term has been reduced by 2). Also, each sequence in U can be obtained in this way (since adding 2 to each term in one of these ES gives a distinct sequence in U).
Therefore, the number of sequences in this case is $e(n-2)$.

This concludes our proof and shows that $e(n) = 1 + e(n-1) + e(n-2)$, as required.