The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Canadian Senior Mathematics Contest

Tuesday, November 20, 2012
(in North America and South America)

Wednesday, November 21, 2012
(outside of North America and South America)

Solutions

Part A

1. The perimeter of $ABCDEF$ equals $AB + BC + CD + DE + EF + FA$.
Since all angles in the diagram are right angles, then AB and EF are parallel to CD.
Also, the sum of their lengths equals the length of CD.
Similarly, AF and ED are parallel to BC and $AF + ED = BC$.
Therefore, the perimeter is

$$\begin{aligned} AB + BC + CD + DE + EF + FA &= (AB + EF + CD) + (BC + ED + AF) \\ &= (AB + EF + (AB + EF)) + (BC + BC) \\ &= 2(AB + EF) + 2(BC) \\ &= 2(8 + 5) + 2(15) \\ &= 26 + 30 \\ &= 56 \end{aligned}$$

Answer: 56

2. *Solution 1*
Since $x^3 - 4x = 0$, then $x(x^2 - 4) = 0$ which is equivalent to $x(x + 2)(x - 2) = 0$.
Therefore, the solutions to the equation are $x = 0, -2, 2$.
These three numbers are a, b and c in some order.
Therefore, $abc = 0(-2)(2) = 0$.
(Note that once we determined that one of the solutions was 0, we did not need to do any more work, since the product of the roots is 0.)

Solution 2
When a cubic equation has the form $x^3 - Sx^2 + Tx - P = 0$, then the product of the solutions equals P.
In this problem, the cubic equation has the form $x^3 - 0x^2 + (-4)x - 0 = 0$.
Since the solutions are a, b and c, then $P = abc$.
But $P = 0$, so $abc = 0$.

Answer: 0

3. *Solution 1*
Using the rules for manipulating exponents,

$$\begin{aligned} 3^x &= 3^{20} \cdot 3^{20} \cdot 3^{18} + 3^{19} \cdot 3^{20} \cdot 3^{19} + 3^{18} \cdot 3^{21} \cdot 3^{19} \\ &= 3^{20+20+18} + 3^{19+20+19} + 3^{18+21+19} \\ &= 3^{58} + 3^{58} + 3^{58} \\ &= 3(3^{58}) \\ &= 3^{1+58} \\ &= 3^{59} \end{aligned}$$

Since $3^x = 3^{59}$, then $x = 59$.

Solution 2

We simplify the right side of the given equation by first removing a common factor which we obtain by taking the smallest of the three exponents of the first factors in each term, the smallest of the three exponents of the second factors, and the smallest of the three exponents of the third factors:

$$\begin{aligned} 3^x &= 3^{20} \cdot 3^{20} \cdot 3^{18} + 3^{19} \cdot 3^{20} \cdot 3^{19} + 3^{18} \cdot 3^{21} \cdot 3^{19} \\ &= 3^{18} \cdot 3^{20} \cdot 3^{18}(3^2 \cdot 3^0 \cdot 3^0 + 3^1 \cdot 3^0 \cdot 3^1 + 3^0 \cdot 3^1 \cdot 3^1) \\ &= 3^{18+20+18}(9 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 3 + 1 \cdot 3 \cdot 3) \\ &= 3^{56}(9+9+9) \\ &= 3^{56}(27) \\ &= 3^{56}3^3 \\ &= 3^{59} \end{aligned}$$

Since $3^x = 3^{59}$, then $x = 59$.

Answer: $x = 59$

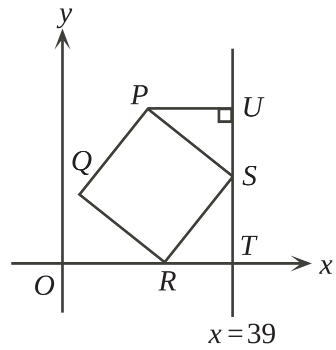
4. Suppose that g is the number of gold pucks. We are told that the first box contains all 40 black pucks and exactly $\frac{1}{7}$ of the gold pucks, so it contains $40 + \frac{1}{7}g$ pucks.
The remaining two boxes must contain the remaining $\frac{6}{7}g$ gold pucks and no black pucks.
We are told that each box contains the same number of pucks.
Since each of the second and third boxes contains the same number of pucks, then each must contain $\frac{1}{2} \times \frac{6}{7}g = \frac{3}{7}g$ gold pucks.
Since the first and second boxes contain the same number of pucks, then $40 + \frac{1}{7}g = \frac{3}{7}g$ or $\frac{2}{7}g = 40$, and so $g = \frac{7}{2}(40) = 140$.
Therefore, there are 140 gold hockey pucks in total.
(We can verify that if there are 140 gold pucks, then the first box contains 40 black pucks and 20 gold pucks (or 60 pucks in total), and the remaining two boxes each contain half of the remaining 120 gold pucks, or 60 gold pucks each in total.)

Answer: 140

5. Since the side length of the square is 25, then $QR = RS = SP = PQ = 25$.
Since Q has coordinates $(0, 7)$, then $QO = 7$.
Since $\triangle QOR$ is right-angled at O, then by the Pythagorean Theorem,

$$OR = \sqrt{QR^2 - OQ^2} = \sqrt{25^2 - 7^2} = \sqrt{576} = 24$$

since $OR > 0$. (We could also recognize the special 7-24-25 right-angled triangle.)
Let T be the point where the line $x = 39$ crosses the x-axis. T has coordinates $(39, 0)$.
Since $OR = 24$, then $RT = OT - OR = 39 - 24 = 15$.
Now rotate the square about R until S lies on the line $x = 39$.
Let U be the point on the line $x = 39$ so that PU is perpendicular to UT. Note that U is above S on the line $x = 39$ since P is above S.



Since $\triangle RTS$ is right-angled at T, then by the Pythagorean Theorem,

$$ST = \sqrt{RS^2 - RT^2} = \sqrt{25^2 - 15^2} = \sqrt{400} = 20$$

since $ST > 0$. (We could also recognize the special 15-20-25 right-angled triangle.)
Since ST is vertical, then S has coordinates $(39, 20)$.
Finally, $\triangle SUP$ is congruent to $\triangle RTS$, because each is right-angled, they have equal hypotenuses, and

$$\angle SPU = 90^\circ - \angle PSU = 90^\circ - (180^\circ - 90^\circ - \angle RST) = \angle RST$$

Therefore, $PU = ST = 20$ and $SU = RT = 15$.
Since S has coordinates $(39, 20)$, then U has coordinates $(39, 20 + 15) = (39, 35)$ and so P has coordinates $(39 - 20, 35) = (19, 35)$.

Answer: $(19, 35)$

6. We picture the floor as having 2 rows and 13 columns.
We solve the problem by determining the number of ways in which we can choose the 11 tiles to be black (and make the remaining tiles white). Since no two black tiles are to be adjacent, then it cannot be the case that both tiles in any column are black.
Thus, each of the 13 columns has 0 black tiles or 1 black tile.
Since there are to be 11 black tiles, then 11 columns will have 1 black tile and 2 columns will have no black tiles.
We denote each column with no black tiles by "E".
If a column has 1 black tile, it is either the top tile (we denote the column "T") or the bottom tile ("B").
Since no two black tiles are adjacent, then there cannot be adjacent columns both having the top tile black or both having the bottom tile black.
So the given problem is equivalent to determining the number of sequences of 13 characters (one for each column), where each character is E, T or B, with exactly two E's (for the empty columns), and with the property that there are not two (or more) consecutive T's or two (or more) consecutive B's.

We count the number of such sequences by considering a number of cases. Before considering these cases, we note that whenever there is a block of consecutive T's and B's, then there are exactly two possible configurations for this block. This is because the first letter (for which there are 2 choices) determines the second letter (since there cannot be consecutive T's or B's) which determines the third letter, and so on.

Case 1: Two E's together; EE at one end of sequence
There are 2 possible locations for the EE (either end of the sequence).
In either case, there is one block of T's and B's.
There are 2 possible configurations for the single resulting block of T's and B's (TBTB$\cdots$ or BTBT$\cdots$).
In this case, there are $2 \times 2 = 4$ sequences.

Case 2: Two E's together; EE not at end of sequence
There are 10 possible locations for the EE (starting in any column from the 2nd column to the 11th column).
In this case, there are two blocks of T's and B's.
There are 2 possible configurations for the each of the two resulting blocks of T's and B's.
In this case, there are $10 \times 2 \times 2 = 40$ sequences.

Case 3: Two E's separate; one at each end of sequence
The position of the E's are fixed.
In this case, there is one block of T's and B's.
There are 2 possible configurations for the single resulting block of T's and B's.
In this case, there are 2 sequences.

Case 4: Two E's separate; one at an end of the sequence, one not at an end
There are 2 possible locations for the E that is at an end of the sequence.
There are 10 possible locations for the E that is not at an end (it cannot be at either end or next to the E that is at an end).
There are two blocks of T's and B's.
There are 2 possible configurations for the each of the two resulting blocks of T's and B's.
In this case, there are $2 \times 10 \times 2 \times 2 = 80$ sequences.

Case 5: Two E's separate; neither at an end of the sequence
The leftmost E could be in column 2. There are then 9 possible locations for the second E (columns 4 through 12).
The leftmost E could be in column 3. There are then 8 possible locations for the second E (columns 5 through 12).
We continue in this way to the last possibility, where the leftmost E is in column 10. There is then 1 possible location for the second E (column 12).
This gives $9 + 8 + 7 + \cdots + 1 = 45$ possible locations for the E's.
There are three blocks of T's and B's (to the left of the first E, between the two E's, and to the right of the second E).
There are 2 possible configurations for the each of the three resulting blocks of T's and B's.
In this case, there are $45 \times 2 \times 2 \times 2 = 360$ sequences.

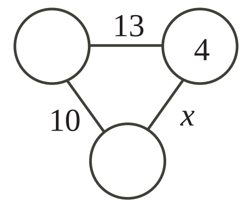
Combining all of the cases, we see that there are $4+40+2+80+360 = 486$ possible sequences, and so 486 possible ways of tiling the floor.

ANSWER: 486

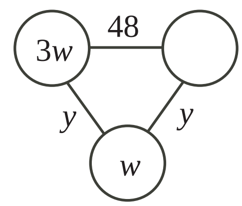
Part B

1. Through this solution we call the number in the top left circle the "left number", the number in the top right circle the "right number", and the number in the bottom circle the "bottom number".
   Further, we call the line joining the left and right numbers the "top line", the line joining the left and bottom numbers the "left line" and the line joining the right and bottom numbers the "right line".

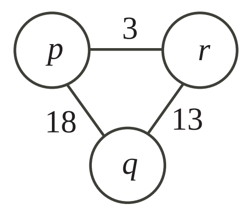
   (a) Since the sum of the numbers along the top line is 13, then the left number is $13 - 4 = 9$.
   Since the sum of the numbers along the left line is 10, then the bottom number is $10 - 9 = 1$.
   Since x is the sum of the numbers along the right line, then $x = 1 + 4 = 5$.



   (b) From the left line, $3w + w = y$ or $y = 4w$.
   Since the sum of the numbers along the right line is also y, then the right number is $y - w = 4w - w = 3w$.
   From the top line, $3w + 3w = 48$ or $6w = 48$ and so $w = 8$.
   Therefore, $y = 4w = 32$.



   (c) From the top line, $p + r = 3$.
   From the left line, $p + q = 18$.
   From the right line, $q + r = 13$.
   There are many different ways of solving this system of three equations in three unknowns. Here are two methods of doing this.



   <u>Method 1</u>
   We add the three equations to obtain $(p + r) + (p + q) + (q + r) = 3 + 18 + 13$ or $2p + 2q + 2r = 34$.
   Dividing by 2, we obtain $p + q + r = 17$.
   Therefore, $p = (p+q+r)-(q+r) = 17-13 = 4$ and $q = (p+q+r)-(p+r) = 17-3 = 14$ and $r = (p + q + r) - (p + q) = 17 - 18 = -1$.
   (We can check these values in the original diagram.)

   <u>Method 2</u>
   From the first equation $r = 3 - p$.
   Substituting this into the third equation, we obtain $q + (3 - p) = 13$ or $q - p = 10$.
   Therefore, we have $p + q = 18$ and $q - p = 10$.
   Adding these two equations, we obtain $2q = 28$ or $q = 14$ and so $p = 4$, whence $r = -1$.

   There are many other ways to solve this system of equations.
   The required values are $p = 4$, $q = 14$ and $r = -1$.

2. (a) With some trial and error, we find that the pair $(x, y) = (3, 2)$ satisfies the given equation, since $3^2 - 2(2^2) = 1$.
If we were asked to justify that this was the only solution or if we wanted a more systematic way to find this pair, we could try the possible values of x from 1 to 5 and try to find one that gives a positive integer value of y.
If $x = 1$, then $1 - 2y^2 = 1$ or $2y^2 = 0$, which does not give any positive integer solutions.
If $x = 2$, then $4 - 2y^2 = 1$ or $2y^2 = 3$, which does not give any positive integer solutions.
If $x = 3$, then $9 - 2y^2 = 1$ or $2y^2 = 8$, which has $y = 2$ as a solution.
If $x = 4$, then $16 - 2y^2 = 1$ or $2y^2 = 15$, which does not give any positive integer solutions.
If $x = 5$, then $25 - 2y^2 = 1$ or $2y^2 = 24$, which does not give any positive integer solutions.
Therefore, the only pair of positive integers that satisfies equation 1 and has $x \leq 5$ is $(x, y) = (3, 2)$.

(b) Expanding $(3 + 2\sqrt{2})^2$, we obtain

$$u + v\sqrt{2} = (3 + 2\sqrt{2})(3 + 2\sqrt{2}) = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 = 17 + 12\sqrt{2}$$

Therefore, $(u, v) = (17, 12)$ satisfies the equation $(3 + 2\sqrt{2})^2 = u + v\sqrt{2}$.
Furthermore, if $(u, v) = (17, 12)$, then $u^2 - 2v^2 = 17^2 - 2(12^2) = 289 - 2(144) = 1$, so $(u, v) = (17, 12)$ satisfies equation 1, as required.

(c) Since (a, b) satisfies 1, then $a^2 - 2b^2 = 1$.
Since $(a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}$, then

$$c + d\sqrt{2} = 3a + 2a\sqrt{2} + 3b\sqrt{2} + 2b(2) = (3a + 4b) + (2a + 3b)\sqrt{2}$$

and so $(c, d) = (3a + 4b, 2a + 3b)$.

(It is not hard to see that *if* $(c, d) = (3a + 4b, 2a + 3b)$, *then* we have

$$c + d\sqrt{2} = (3a + 4b) + (2a + 3b)\sqrt{2}$$

To formally justify that $c+d\sqrt{2} = (3a+4b)+(2a+3b)\sqrt{2}$ implies $(c, d) = (3a+4b, 2a+3b)$, we can rearrange the equation to obtain

$$c - 3a - 4b = (2a + 3b - d)\sqrt{2}$$

The left side of this equation is an integer. If $2a+3b-d \neq 0$, the right side of this equation is irrational (because $\sqrt{2}$ is irrational), so cannot equal an integer. Thus, $2a + 3b - d = 0$ or $d = 2a + 3b$. This implies that $c - 3a - 4b = 0$ or $c = 3a + 4b$.)

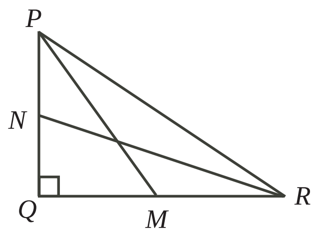
To show that (c, d) satisfies equation 1, we need to show that $c^2 - 2d^2 = 1$:

$$\begin{aligned}
c^2 - 2d^2 &= (3a + 4b)^2 - 2(2a + 3b)^2 \\
&= (9a^2 + 24ab + 16b^2) - 2(4a^2 + 12ab + 9b^2) \\
&= 9a^2 + 24ab + 16b^2 - 8a^2 - 24ab - 18b^2 \\
&= a^2 - 2b^2 \\
&= 1 \qquad (\text{since } a^2 - 2b^2 = 1)
\end{aligned}$$

Therefore, (c, d) satisfies equation 1, as required.

(d) From (c), we know that if (a, b) is a solution to equation ①, then $(c, d) = (3a+4b, 2a+3b)$ is also a solution to equation ①.
From (b), we know that $(17, 12)$ is a solution to equation ①.
We use these two facts together.
Since $(17, 12)$ is a solution, then $(3(17) + 4(12), 2(17) + 3(12)) = (99, 70)$ is a solution.
Since $(99, 70)$ is a solution, then $(3(99) + 4(70), 2(99) + 3(70)) = (577, 408)$ is a solution, which has $y > 100$.
We can verify that $577^2 - 2(408^2) = 1$, as required.

3. (a) Since $PQ = 6$ and N is the midpoint of PQ, then $PN = NQ = 3$.
Since $QR = 8$ and M is the midpoint of QR, then $QM = MR = 4$.



Since $\triangle PQM$ is right-angled at Q and $PM > 0$, we can apply the Pythagorean Theorem to obtain

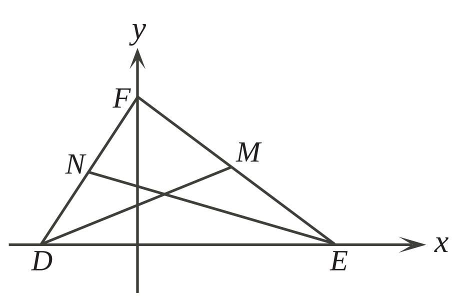
$$PM = \sqrt{PQ^2 + QM^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$

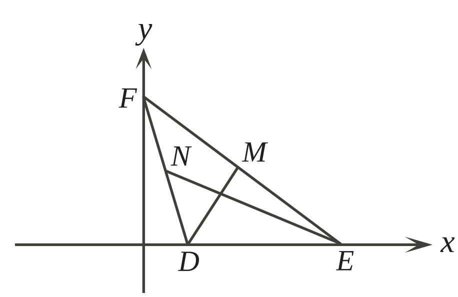
Since $\triangle NQR$ is right-angled at Q and $RN > 0$, we can apply the Pythagorean Theorem to obtain

$$RN = \sqrt{NQ^2 + QR^2} = \sqrt{3^2 + 8^2} = \sqrt{73}$$

Therefore, the two medians have lengths $PM = 2\sqrt{13}$ and $RN = \sqrt{73}$.

(b) *Solution 1*
Suppose that DM and EN are the two medians of $\triangle DEF$ that are equal in length.
Place $\triangle DEF$ in the coordinate plane so that DE lies along the x-axis and so that F is on the positive y-axis. (We can do this by rotating and translating the triangle as necessary.)
We may assume that the rightmost of D and E is on the positive x-axis (since we can reflect the triangle in the y-axis if necessary), and that it is vertex E that is the rightmost of D and E (since we can switch the labels if necessary).
Give F coordinates $(0, 2c)$, E coordinates $(2b, 0)$, and D coordinates $(2a, 0)$ for some real numbers a, b, c with $b > a$ and $b > 0$ and $c > 0$. The number a might be positive, negative or zero. (The cases $a < 0$ and $a > 0$ are illustrated below.)





Since DM and EN are medians, then M is the midpoint of EF and N is the midpoint of DF.
Since E has coordinates $(2b, 0)$ and F has coordinates $(0, 2c)$, then M has coordinates $(\frac{1}{2}(2b + 0), \frac{1}{2}(0 + 2c)) = (b, c)$.
Since D has coordinates $(2a, 0)$ and F has coordinates $(0, 2c)$, then N has coordinates $(\frac{1}{2}(2a + 0), \frac{1}{2}(0 + 2c)) = (a, c)$.
Since D has coordinates $(2a, 0)$ and M has coordinates (b, c), then

$$DM = \sqrt{(2a - b)^2 + (0 - c)^2} = \sqrt{(2a - b)^2 + c^2}$$

Since E has coordinates $(2b, 0)$ and N has coordinates (a, c), then

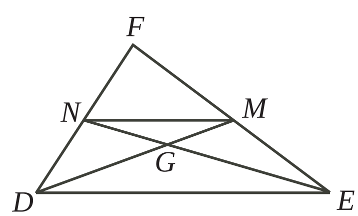
$$EN = \sqrt{(2b - a)^2 + (0 - c)^2} = \sqrt{(2b - a)^2 + c^2}$$

Since DM and EN are equal in length, then

$$\begin{aligned} DM^2 &= EN^2 \\ (2a - b)^2 + c^2 &= (2b - a)^2 + c^2 \\ (2a - b)^2 &= (2b - a)^2 \\ 4a^2 - 4ab + b^2 &= 4b^2 - 4ab + a^2 \\ 3a^2 &= 3b^2 \\ a^2 &= b^2 \end{aligned}$$

Therefore, $a = \pm b$.
Since D and E are distinct points, then $a \neq b$.
Thus, $a = -b$.
Therefore, the vertices of $\triangle DEF$ are $D(-2b, 0)$, $E(2b, 0)$, and $F(0, 2c)$.
But then sides DF and EF are equal in length (each has length $\sqrt{4b^2 + 4c^2}$), which means that $\triangle DEF$ is isosceles, as required.

Solution 2
Suppose that DM and EN are the two medians of $\triangle DEF$ that are equal in length, and that they cross at G.
Join M to N.

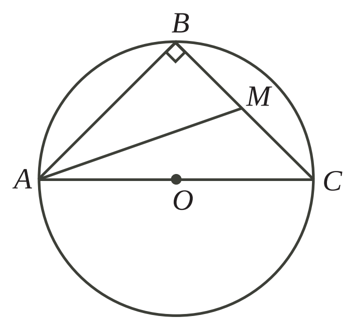


Since DM and EN are medians, then M is the midpoint of FE (which gives $FM = ME$) and N is the midpoint of FD (which gives $FN = ND$).
Since $\dfrac{FD}{FN} = \dfrac{FE}{FM} = 2$ and the triangles share a common angle at F, then $\triangle FDE$ and $\triangle FNM$ are similar, with similarity ratio $2 : 1$. This tells us that $\angle FDE = \angle FNM$.
Thus, NM is parallel to DE.
Also, $DE = 2NM$.
Consider next $\triangle GNM$ and $\triangle GED$.

Since NM is parallel to DE, then $\angle GNM = \angle GED$ and $\angle GMN = \angle GDE$.
This means that $\triangle GNM$ is similar to $\triangle GED$. Since $NM = \frac{1}{2}DE$, then the similarity ratio is $1 : 2$.
Therefore, $GE = 2GN$ and $GD = 2GM$.
Since $DM = EN$, then $GD + GM = GE + GN$ or $2GM + GM = 2GN + GN$. This gives $3GM = 3GN$ or $GM = GN$.
Therefore, $\triangle GNM$ is isosceles with $\angle GNM = \angle GMN$.
Finally, consider $\triangle MNE$ and $\triangle NMD$.
Since $\angle MNE = \angle NMD$, NM is a common side, and $EN = DM$, then the triangles are congruent (SAS).
Thus, $ME = ND$.
Since $ME = \frac{1}{2}FE$ and $ND = \frac{1}{2}FD$, then $FE = FD$, and so $\triangle DEF$ is isosceles, as required.

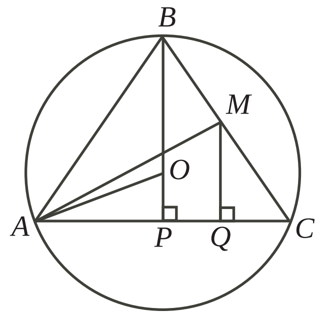
(c) Suppose that $\triangle ABC$ has equal medians $AM = CN$, with M the midpoint of BC and N the midpoint of AB. Suppose also that O is the centre of the circle.
We are given that $AM = CN = r$.
From (b), $\triangle ABC$ must be isosceles with $AB = BC$.
We consider three cases: $\angle ABC = 90^\circ$, $\angle ABC < 90^\circ$, and $\angle ABC > 90^\circ$.

Case 1: $\angle ABC = 90^\circ$
Since $\angle ABC = 90^\circ$, then AC is a diameter of the circle with centre O and radius r.
Thus, $AC = 2r$.
Further, since $\triangle ABC$ is right-angled and isosceles, then $AB = BC = \frac{1}{\sqrt{2}}AC = \sqrt{2}r$.
We draw median AM, which has length r.



Now $\triangle MBA$ is right-angled at B, which means that AM is its longest side.
Therefore, $AM > AB = \sqrt{2}r$.
But $AM = r$, which is a contradiction.
Therefore, $\angle ABC \neq 90^\circ$.

Case 2: $\angle ABC < 90^\circ$
Here, $\angle ABC$ is acute and $AB = BC$.
Since $\angle ABC$ is acute, then B is on the opposite side of O as AC.
Let P be the midpoint of AC. Draw median BP.
Since $\triangle ABC$ is isosceles, then BP is perpendicular to AC at P.
Since O is the centre of the circle and AC is a chord with midpoint P, then OP is perpendicular to AC at P.
Since BP and OP are each perpendicular to AC at P, then BP passes through O.
Draw a perpendicular from M to Q on AC.

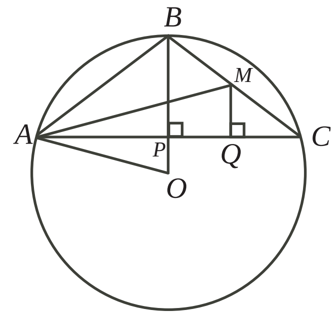


Suppose that $OP = x$.
Since $OA = r$ (it is a radius) and $OP = x$, then $AP = \sqrt{OA^2 - OP^2} = \sqrt{r^2 - x^2}$.
Now $\triangle MQC$ is similar to $\triangle BPC$, since each is right-angled and they share a common angle at C.
Since $BC = 2MC$, then $PC = 2QC$ and $BP = 2MQ$.
Since $QC = \frac{1}{2}PC$, then $PQ = PC - QC = \frac{1}{2}PC = \frac{1}{2}AP = \frac{1}{2}\sqrt{r^2 - x^2}$.
Since $BO = r$ and $OP = x$, then $MQ = \frac{1}{2}BP = \frac{1}{2}(r + x)$.
Finally, $\triangle AMQ$ is right-angled at Q, so by the Pythagorean Theorem, we have

$$\begin{aligned}
AQ^2 + MQ^2 &= AM^2 \\
(AP + PQ)^2 + MQ^2 &= AM^2 \\
\left(\sqrt{r^2 - x^2} + \tfrac{1}{2}\sqrt{r^2 - x^2}\right)^2 + \left(\tfrac{1}{2}(r + x)\right)^2 &= r^2 \\
\left(\tfrac{3}{2}\sqrt{r^2 - x^2}\right)^2 + \left(\tfrac{1}{2}(r + x)\right)^2 &= r^2 \\
\tfrac{9}{4}(r^2 - x^2) + \tfrac{1}{4}(r^2 + 2rx + x^2) &= r^2 \\
9(r^2 - x^2) + (r^2 + 2rx + x^2) &= 4r^2 \\
6r^2 + 2rx - 8x^2 &= 0 \\
3r^2 + rx - 4x^2 &= 0 \\
(3r + 4x)(r - x) &= 0
\end{aligned}$$

Therefore, $3r = -4x$ (which is impossible since r and x are both positive) or $r = x$ (which is also impossible since this would mean that $AP = PC = \sqrt{r^2 - r^2} = 0$).
Since neither result is possible, then it cannot be the case that $\angle ABC < 90^\circ$.

Case 3: $\angle ABC > 90^\circ$
Here, $\angle ABC$ is obtuse and $AB = BC$.
Since $\angle ABC$ is obtuse, then B is on the same side of O as AC.
Let P be the midpoint of AC.
Draw BP and OP.
Since $\triangle ABC$ is isosceles, then BP is perpendicular to AC.
Since P is the midpoint of AC, then OP is perpendicular to AC as well.
Therefore, BPO is a straight line segment.
Draw a perpendicular from M to Q on AC.



Suppose that $OP = x$.
Since $OA = r$ (it is a radius) and $OP = x$, then $AP = \sqrt{OA^2 - OP^2} = \sqrt{r^2 - x^2}$.
Now $\triangle MQC$ is similar to $\triangle BPC$, since each is right-angled and they share a common angle at C.
Since $BC = 2MC$, then $PC = 2QC$ and $BP = 2MQ$.
Since $QC = \frac{1}{2}PC$, then $PQ = PC - QC = \frac{1}{2}PC = \frac{1}{2}AP = \frac{1}{2}\sqrt{r^2 - x^2}$.
Since $BO = r$ and $OP = x$, then $MQ = \frac{1}{2}BP = \frac{1}{2}(r - x)$.
Finally, $\triangle AMQ$ is right-angled at Q, so by the Pythagorean Theorem, we have

$$\begin{aligned}
AQ^2 + MQ^2 &= AM^2 \\
(AP + PQ)^2 + MQ^2 &= AM^2 \\
\left(\sqrt{r^2 - x^2} + \tfrac{1}{2}\sqrt{r^2 - x^2}\right)^2 + \left(\tfrac{1}{2}(r - x)\right)^2 &= r^2 \\
\left(\tfrac{3}{2}\sqrt{r^2 - x^2}\right)^2 + \left(\tfrac{1}{2}(r - x)\right)^2 &= r^2 \\
\tfrac{9}{4}(r^2 - x^2) + \tfrac{1}{4}(r^2 - 2rx + x^2) &= r^2 \\
9(r^2 - x^2) + (r^2 - 2rx + x^2) &= 4r^2 \\
6r^2 - 2rx - 8x^2 &= 0 \\
3r^2 - rx - 4x^2 &= 0 \\
(3r - 4x)(r + x) &= 0
\end{aligned}$$

Therefore, $r = -x$ (which is impossible since r and x are both positive) or $3r = 4x$.
In this last case, $x = \frac{3}{4}r$.
Then $AC = 2\sqrt{r^2 - x^2} = 2\sqrt{r^2 - \frac{9}{16}r^2} = 2\sqrt{\frac{7}{16}r^2} = \frac{2\sqrt{7}}{4}r = \frac{\sqrt{7}}{2}r$.
Also, $AB = BC = \sqrt{AP^2 + BP^2} = \sqrt{\left(\sqrt{\frac{7}{16}r^2}\right)^2 + \left(r - \frac{3}{4}r\right)^2} = \sqrt{\frac{7}{16}r^2 + \frac{1}{16}r^2} = \frac{\sqrt{2}}{2}r$.

Therefore, having examined all of the possible cases, the side lengths of $\triangle ABC$ must be $AC = \frac{\sqrt{7}}{2}r$ and $AB = BC = \frac{\sqrt{2}}{2}r$.