The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Canadian Intermediate Mathematics Contest

Tuesday, November 20, 2012
(in North America and South America)

Wednesday, November 21, 2012
(outside of North America and South America)

Solutions

Part A

1. The line consists of Jeanne herself, the four people in front of Jeanne, and the seven people behind her. Therefore, there are $4 + 1 + 7 = 12$ people in the line.

Answer: 12

2. Since $1^{20} = 1$ and each of the other numbers is larger than 1, then 1^{20} is not the largest of the numbers.
We convert the last three numbers to powers of 2, using the rule that $(2^a)^b = 2^{ab}$:

$$4^8 = (2^2)^8 = 2^{16} \qquad 8^5 = (2^3)^5 = 2^{15} \qquad 16^3 = (2^4)^3 = 2^{12}$$

Therefore, the numbers in the list are $1, 2^{14}, 2^{16}, 2^{15}, 2^{12}$.
Since the bases of the last four numbers are equal and this base is larger than 1, then the largest number is the one with the largest exponent.
Therefore, $4^8 = 2^{16}$ is the largest number in the list.
(We could have used a calculator to calculate the five numbers to be $1, 16\,384, 65\,536, 32\,764, 4096$.)

Answer: $4^8 = 2^{16} = 65\,536$

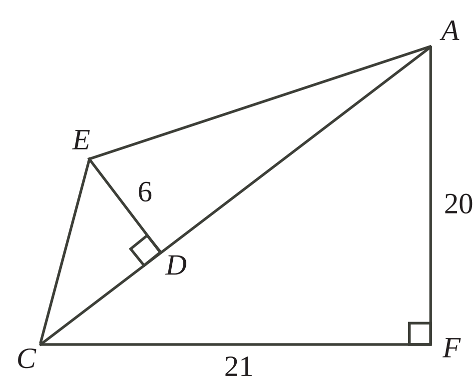
3. Let the length of the original rectangle be l and the width of the original rectangle be w.
Since the length is three times the width, then $l = 3w$.
We are also told that if the length is decreased by 5 and the width is increased by 5, then the rectangle becomes square. This means that $l - 5 = w + 5$.
Substituting the first equation into the second, we obtain $3w - 5 = w + 5$ or $2w = 10$, from which $w = 5$.
Since $l = 3w$, then $l = 3(5) = 15$; that is, the length of the original rectangle is 15.

Answer: 15

4. The area of quadrilateral $AFCE$ equals the sum of the areas of $\triangle AFC$ and $\triangle ACE$.
Since $\triangle AFC$ is right-angled at F, then its area equals $\frac{1}{2}(AF)(CF) = \frac{1}{2}(20)(21) = 210$.
Since $\triangle AFC$ is right-angled at F, then we can use the Pythagorean Theorem to obtain

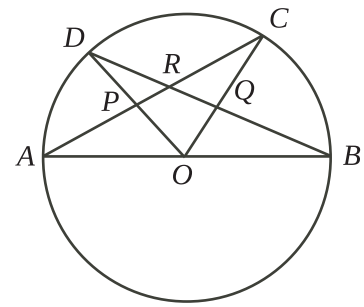
$$AC = \sqrt{AF^2 + CF^2} = \sqrt{20^2 + 21^2} = \sqrt{841} = 29$$

since $AC > 0$.
Since ED is perpendicular to AC, then $\triangle ACE$ can be viewed as having base AC and height ED.
Therefore, the area of $\triangle ACE$ is $\frac{1}{2}(AC)(DE)$ or $\frac{1}{2}(29)(6)$ or 87.
Thus, the area of quadrilateral $AFCE$ is $210 + 87 = 297$.



Answer: 297

5. *Solution 1*



Since O is the centre of the circle, then $OA = OC$ and $OB = OD$.
This means that each of $\triangle OAC$ and $\triangle OBD$ is isosceles.
Therefore, $\angle OAC = \angle OCA$ and $\angle OBD = \angle ODB$.
Since $\angle BOQ = 60^\circ$, then $\angle AOC = 180^\circ - \angle BOQ = 180^\circ - 60^\circ = 120^\circ$.
Since the angles in $\triangle OAC$ add to 180° and $\angle OAC = \angle OCA$, then

$$\angle OAC = \tfrac{1}{2}(180^\circ - \angle AOC) = \tfrac{1}{2}(180^\circ - 120^\circ) = 30^\circ$$

Next, consider $\triangle APO$.
We have $\angle PAO = \angle OAC = 30^\circ$ and $\angle APO = 100^\circ$.
Therefore, $\angle AOP = 180^\circ - 100^\circ - 30^\circ = 50^\circ$.
Then $\angle BOD = 180^\circ - \angle AOP = 180^\circ - 50^\circ = 130^\circ$.
Since the angles in $\triangle OBD$ add to 180° and $\angle OBD = \angle ODB$, then

$$\angle OBD = \tfrac{1}{2}(180^\circ - \angle BOD) = \tfrac{1}{2}(180^\circ - 130^\circ) = 25^\circ$$

Finally, we consider $\triangle BQO$.
We have $\angle QBO = \angle OBD = 25^\circ$ and $\angle BOQ = 60^\circ$.
Therefore, $\angle BQO = 180^\circ - 25^\circ - 60^\circ = 95^\circ$.

Solution 2

Since O is the centre of the circle, then $OA = OC$ and $OB = OD$.
This means that each of $\triangle OAC$ and $\triangle OBD$ is isosceles.
Therefore, $\angle OAC = \angle OCA = x^\circ$ and $\angle OBD = \angle ODB = y^\circ$ for some numbers x and y.
Since $\angle BOQ$ is an exterior angle to $\triangle OAC$, then $\angle OAC + \angle OCA = \angle BOQ$, from which we obtain $x^\circ + x^\circ = 60^\circ$ or $2x = 60$ or $x = 30$.
Next, consider $\triangle APO$.
We have $\angle PAO = \angle OAC = 30^\circ$ and $\angle APO = 100^\circ$.
Therefore, $\angle AOP = 180^\circ - 100^\circ - 30^\circ = 50^\circ$.
Since $\angle AOP$ is an exterior angle to $\triangle ODB$, then $\angle ODB + \angle OBD = \angle AOP$, from which we obtain $y^\circ + y^\circ = 50^\circ$ or $2y = 50$ or $y = 25$.
Finally, we consider $\triangle BQO$.
We have $\angle QBO = \angle OBD = 25^\circ$ and $\angle BOQ = 60^\circ$.
Therefore, $\angle BQO = 180^\circ - 25^\circ - 60^\circ = 95^\circ$.

Solution 3

Since O is the centre of the circle, then $\angle BOC = 2\angle BAC$. (This uses the property of circles that the angle formed by a chord at the centre of the circle is twice the angle formed on the major arc.) Since $\angle BOC = \angle BOQ = 60^\circ$, then $\angle BAC = 30^\circ$.
Since O is the centre of the circle, then $OA = OC$ and $OB = OD$.
This means that each of $\triangle OAC$ and $\triangle OBD$ is isosceles.
Next, consider $\triangle APO$.
We have $\angle PAO = \angle BAC = 30^\circ$ and $\angle APO = 100^\circ$.
Therefore, $\angle AOP = 180^\circ - 100^\circ - 30^\circ = 50^\circ$.
Using the same circle property as above, $\angle ABD = \tfrac{1}{2}\angle AOD = \tfrac{1}{2}\angle AOP = 25^\circ$.
Finally, we consider $\triangle BQO$.
We have $\angle QBO = \angle ABD = 25^\circ$ and $\angle BOQ = 60^\circ$.
Therefore, $\angle BQO = 180^\circ - 25^\circ - 60^\circ = 95^\circ$.

ANSWER: 95°

6. Since we are told that $(xyz)_b = xb^2 + yb + z$, then $(xyz)_{10} = 10^2x + 10y + z = 100x + 10y + z$ and $(xyz)_7 = 7^2x + 7y + z = 49x + 7y + z$.
From the given information,

$$\begin{aligned} (xyz)_{10} &= 2(xyz)_7 \\ 100x + 10y + z &= 2(49x + 7y + z) \\ 100x + 10y + z &= 98x + 14y + 2z \\ 2x &= 4y + z \end{aligned}$$

Since the left side of this equation is an even integer $(2x)$, then the right side must also be an even integer.
Since $4y$ is an even integer, then for $4y + z$ to be an even integer, it must be the case that z is an even integer.
This gives us three possibilities: $z = 2$, $z = 4$ and $z = 6$.

Case 1: $z = 2$
Here, $2x = 4y + 2$ or $x = 2y + 1$.
We try the possible values for y:

- $y = 1$ gives $x = 2(1) + 1 = 3$; this gives the triple $(x, y, z) = (3, 1, 2)$
- $y = 2$ gives $x = 2(2) + 1 = 5$; this gives the triple $(x, y, z) = (5, 2, 2)$
- If y is at least 3, then $x = 2y + 1$ is at least 7, which is impossible.

Therefore, there are two triples that work when $z = 2$.

Case 2: $z = 4$
Here, $2x = 4y + 4$ or $x = 2y + 2$.
We try the possible values for y:

- $y = 1$ gives $x = 2(1) + 2 = 4$; this gives the triple $(x, y, z) = (4, 1, 4)$
- $y = 2$ gives $x = 2(2) + 2 = 6$; this gives the triple $(x, y, z) = (6, 2, 4)$
- If y is at least 3, then $x = 2y + 2$ is at least 8, which is impossible.

Therefore, there are two triples that work when $z = 4$.

Case 3: $z = 6$
Here, $2x = 4y + 6$ or $x = 2y + 3$.
We try the possible values for y:

- $y = 1$ gives $x = 2(1) + 3 = 5$; this gives the triple $(x, y, z) = (5, 1, 6)$
- If y is at least 2, then $x = 2y + 3$ is at least 7, which is impossible.

Therefore, there is one triple that works when $z = 6$.

Finally, the triples that satisfy the equation are $(x, y, z) = (3, 1, 2), (5, 2, 2), (4, 1, 4), (6, 2, 4), (5, 1, 6)$.

ANSWER: $(x, y, z) = (3, 1, 2), (5, 2, 2), (4, 1, 4), (6, 2, 4), (5, 1, 6)$

Part B

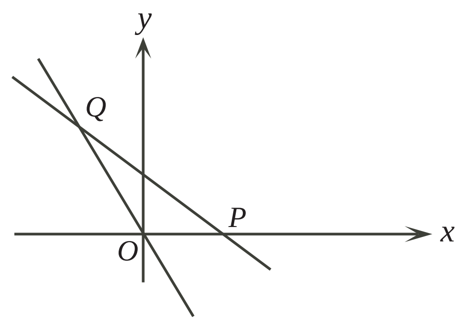
1. (a) A 7 by 7 square has seven columns, and the fourth column is the middle column since it has three columns to the left and three columns to the right.
Rowan shades 2 squares in each of the first, second and third columns, he shades 1 square in the fourth (middle) column, and he shades 2 squares in each of the the fifth, sixth and seventh columns.
In total, he shades $2 + 2 + 2 + 1 + 2 + 2 + 2 = 13$ squares.

(b) In a 101 by 101 square, each diagonal contains 101 squares. The two diagonals overlap in one square (the middle square), which Rowan does not shade twice.
When Rowan shades the first diagonal, he shades 101 squares.
Therefore, when Rowan shades the second diagonal, he shades $101 - 1 = 100$ squares.
In total, he shades $101 + 100 = 201$ squares.

(c) Suppose that the grid that Rowan is given in this part is n by n, with n odd.
We can use the analysis from (b) to conclude that Rowan shades $n + (n - 1) = 2n - 1$ squares.
We are told that Rowan shades 41 squares, so $2n-1 = 41$ or $2n = 42$, which gives $n = 21$.
Now the square contains $n^2 = 21^2 = 441$ squares in total, and Rowan shades 41 of them, so there are $441 - 41 = 400$ unshaded squares.

(d) Suppose that Rowan is given an m by m square with m odd.
The grid contains m^2 squares in total, of which he shades $2m - 1$.
This means that there are $m^2 - (2m - 1) = m^2 - 2m + 1$ unshaded squares in total.
Since we are told that there are 196 unshaded squares in total, then $m^2 - 2m + 1 = 196$.
Now $(m - 1)^2 = m^2 - 2m + 1$, so $(m - 1)^2 = 196$.
Since m is positive, then $m - 1 = \sqrt{196} = 14$ and so $m = 15$.
Therefore, the grid that Rowan is given is 15 by 15, and so contains $15^2 = 225$ squares in total.

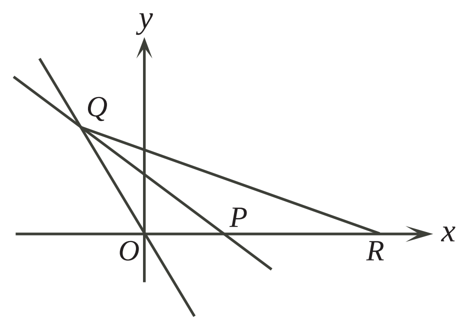
2. (a) To find the coordinates of P, the point where line L_2 crosses the x-axis, we set $y = 0$ and solve for x.
Thus, we solve $0 = -\frac{1}{2}x + 5$ or $\frac{1}{2}x = 5$ to get $x = 10$.
Therefore, the coordinates of P are $(10, 0)$.



To find the coordinates of Q, the point of intersection of lines L_1 and L_2, we equate values of y and solve for x.
Thus, we solve $-\frac{4}{3}x = -\frac{1}{2}x+5$. We multiply both sides by 6 to obtain $6(-\frac{4}{3}x) = 6(-\frac{1}{2}x+5)$ or $-8x = -3x + 30$, which gives $-5x = 30$ or $x = -6$.
To find the y-coordinate of Q, we substitute $x = -6$ into the equation of line L_1 to obtain $y = -\frac{4}{3}(-6) = 8$.
Therefore, the coordinates of Q are $(-6, 8)$.

(b) The coordinates of the vertices of $\triangle OPQ$ are $O(0,0)$, $P(10,0)$, and $Q(-6,8)$.
We can view this triangle as having base OP along the x-axis. OP has length 10.
The height of this triangle is then the distance from Q to the x-axis, or 8.
Therefore, the area $\triangle OPQ$ is $\frac{1}{2}(10)(8) = 40$.

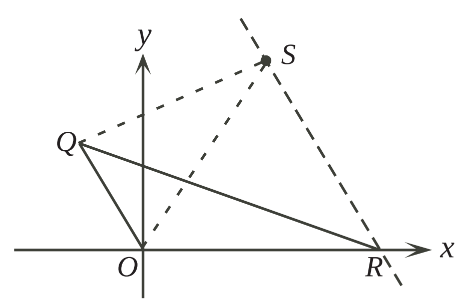
(c) Since the area of $\triangle OPQ$ is 40 and the area of $\triangle OQR$ is three times the area of $\triangle OPQ$, then the area of $\triangle OQR$ is $3 \times 40 = 120$.
Since R is to be on the positive x-axis, then we can view $\triangle OQR$ as having base OR along the x-axis. Suppose that the length of the base is b.



The height of this triangle is again the distance from Q to the x-axis, or 8.
Thus, we want $\frac{1}{2}b(8) = 120$ or $4b = 120$, and so $b = 30$.
Since O has coordinates $(0,0)$ and R is on the positive x-axis, then R must have coordinates $(30,0)$.
(We could also have noted that $\triangle OQR$ and $\triangle OPQ$ can be viewed as having the same height, so for the areas to be in the ratio $3:1$, then the ratio of the lengths of the bases must also be $3:1$.)

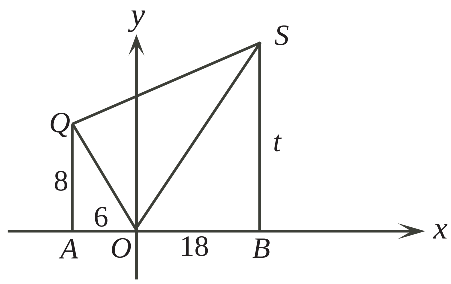
(d) Since the area of $\triangle OQS$ is to be three times the area of $\triangle OPQ$, then its area is to be 120, which is equal to the area of $\triangle OQR$ from (c). Here are two methods to find the value of t.

Method 1: Constant heights
Consider $\triangle OQS$ and $\triangle OQR$.
These triangles are to have the same area, and can be viewed as having a common base, OQ.
Therefore, the two triangles must have the same height.
This height will be the perpendicular distance from $R(30,0)$ to the line $y = -\frac{4}{3}x$, along which OQ lies.
We find the coordinates of S (knowing that it must have an x-coordinate of 18) by sliding a point from R along a line parallel to the base OQ (so along a line with slope $-\frac{4}{3}$). This process will maintain the height.



To get from R to S, we slide $30 - 18 = 12$ units to the left. Since the slope of the line is $-\frac{4}{3}$, then we must go up by $\frac{4}{3}(12) = 16$ units.
Therefore, S has coordinates $(18, 16)$; that is, $t = 16$.

Method 2: Using a trapezoid
Join points O, Q and S to each other, and draw perpendiculars from Q to $A(-6, 0)$ and from S to $B(18, 0)$ on the x-axis.



Then the area of trapezoid $AQSB$ will equal the sum of the areas of $\triangle AQO$, $\triangle OQS$ and $\triangle OSB$.
$\triangle AQO$ has base 6 and height 8 (since the coordinates of Q are $(-6, 8)$) and so $\triangle AQO$ has area $\frac{1}{2}(6)(8) = 24$.
$\triangle OQS$ is to have area 120.
$\triangle OSB$ has base 18 and height t (since the coordinates of S are $(18, t)$) and so $\triangle OSB$ has area $\frac{1}{2}(18)t = 9t$.
Thus, trapezoid $AQSB$ has area $24 + 120 + 9t$.
Viewed another way, trapezoid $AQSB$ has parallel sides AQ and BS of lengths 8 and t, and height AB of length $6 + 18 = 24$, and so has area $\frac{1}{2}(8 + t)(24) = 96 + 12t$.
Equating these areas, we obtain $96 + 12t = 24 + 120 + 9t$ and so $3t = 48$ or $t = 16$.

3. (a) Since the fourth number in the chain is $\frac{2}{11}$ and $\frac{2}{11}$ is less than $\frac{1}{2}$, then the fifth number is $2(\frac{2}{11}) = \frac{4}{11}$.
Since the fifth number in the chain is $\frac{4}{11}$ and $\frac{4}{11}$ is less than $\frac{1}{2}$, then the sixth number is $2(\frac{4}{11}) = \frac{8}{11}$.
Since the sixth number in the chain is $\frac{8}{11}$ and $\frac{8}{11}$ is larger than $\frac{1}{2}$, then the seventh number is $2(1 - \frac{8}{11}) = 2(\frac{3}{11}) = \frac{6}{11}$.
Since the seventh number in the chain is $\frac{6}{11}$ and $\frac{6}{11}$ is larger than $\frac{1}{2}$, then the eighth number is $2(1 - \frac{6}{11}) = 2(\frac{5}{11}) = \frac{10}{11}$.
Therefore, the next four numbers in the chain are $\frac{4}{11}, \frac{8}{11}, \frac{6}{11}, \frac{10}{11}$.

(b) There are two possibilities: $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$.
If $x \leq \frac{1}{2}$ and x is entered into the machine, then the machine outputs $2x$.
Since the input is to be the same as the output, then $x = 2x$ or $x = 0$.
This is impossible since $x > 0$.
If $x > \frac{1}{2}$ and x is entered into the machine, then the machine outputs $2(1 - x)$.
Since the input is to be the same as the output, then $x = 2(1 - x)$ or $x = 2 - 2x$.
Thus, $3x = 2$ and so $x = \frac{2}{3}$, which satisfies the restrictions.
Therefore, if x is entered into the machine and x is produced, then $x = \frac{2}{3}$.

(c) Suppose that the chain is $a, b, c, 1$.
In this part, we have to produce the chain backwards (that is, we have to determine c, b and a).
We make the following general observation:

> Suppose that x is entered into the machine and y is produced.
> If $x \leq \frac{1}{2}$, then $y = 2x$. Solving for x in terms of y, we obtain $x = \frac{1}{2}y$. (This gives us the input in terms of the output.)
> If $x > \frac{1}{2}$, then $y = 2(1-x)$. Solving for x in terms of y, we obtain $y = 2 - 2x$ or $2x = 2 - y$ or $x = \frac{1}{2}(2-y)$.

Since the third number in the chain is c and the fourth is 1, then from above, $c = \frac{1}{2}(1) = \frac{1}{2}$ or $c = \frac{1}{2}(2-1) = \frac{1}{2}$.
In either case, the chain is $a, b, \frac{1}{2}, 1$.
Since the second number in the chain is b and the third is $\frac{1}{2}$, then from above, $b = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ or $b = \frac{1}{2}(2 - \frac{1}{2}) = \frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$.
Therefore, the chain is either $a, \frac{1}{4}, \frac{1}{2}, 1$ or $a, \frac{3}{4}, \frac{1}{2}, 1$.

If the chain is $a, \frac{1}{4}, \frac{1}{2}, 1$, then the first number is a and the second number is $\frac{1}{4}$.
Thus, either $a = \frac{1}{2}(\frac{1}{4}) = \frac{1}{8}$ or $a = \frac{1}{2}(2 - \frac{1}{4}) = \frac{1}{2}(\frac{7}{4}) = \frac{7}{8}$.

If the chain is $a, \frac{3}{4}, \frac{1}{2}, 1$, then the first number is a and the second number is $\frac{3}{4}$.
Thus, either $a = \frac{1}{2}(\frac{3}{4}) = \frac{3}{8}$ or $a = \frac{1}{2}(2 - \frac{3}{4}) = \frac{1}{2}(\frac{5}{4}) = \frac{5}{8}$.

Therefore, the possible first numbers in the chain are $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$.
We can double check that each of these gives a fourth number of 1:

$$\tfrac{1}{8} \to \tfrac{1}{4} \to \tfrac{1}{2} \to 1 \qquad \tfrac{3}{8} \to \tfrac{3}{4} \to \tfrac{1}{2} \to 1 \qquad \tfrac{5}{8} \to \tfrac{3}{4} \to \tfrac{1}{2} \to 1 \qquad \tfrac{7}{8} \to \tfrac{1}{4} \to \tfrac{1}{2} \to 1$$

(d) Suppose that the first number in the chain is $x = \dfrac{2}{m}$ for some positive integer m.
We want the eighth number in the chain also to equal x.
We call the step $y \to 2y$ "A" and the step $y \to 2(1-y)$ "B".
It takes seven steps to get from the first number to the eighth number.
We try to find three integers m that satisfy the requirements by actually building three different chains. We start with the idea that m might be large and so each of the first several steps applied is the doubling step (A).
Note that if the chain were produced by applying steps AAAAAAA in this order, then the chain would be

$$x \to 2x \to 4x \to 8x \to 16x \to 32x \to 64x \to 128x$$

and so for the eighth number to equal the first, we would have $x = 128x$ or $127x = 0$, and so $x = 0$, which is not possible.

Suppose that the chain is produced by applying steps AAAAAAB in this order.
Then the chain is

$$x \to 2x \to 4x \to 8x \to 16x \to 32x \to 64x \to 2(1-64x)$$

Since the eighth number equals the first, then $x = 2(1-64x)$ or $x = 2 - 128x$, and so $129x = 2$ or $x = \dfrac{2}{129}$.

Thus, $m = 129$ is a possible value of m.
We double check this possible value by writing out the resulting chain:

$$\frac{2}{129} \to \frac{4}{129} \to \frac{8}{129} \to \frac{16}{129} \to \frac{32}{129} \to \frac{64}{129} \to \frac{128}{129} \to \frac{2}{129}$$

(Note that the first six numbers are less than $\frac{1}{2}$ so step A is applied to double the number; the seventh number is greater than $\frac{1}{2}$ so step B is applied.)

Suppose next that the chain is produced by applying steps AAAAABB in this order.
Then the chain is

$$x \to 2x \to 4x \to 8x \to 16x \to 32x \to 2(1-32x) = 2-64x \to 2(1-(2-64x))$$

Since the eighth number equals the first, then $x = 2(1-(2-64x))$ or $x = 2(64x-1)$ or $x = 128x - 2$, and so $127x = 2$ or $x = \dfrac{2}{127}$.
Thus, $m = 127$ is a possible value of m.
We double check this possible value by writing out the resulting chain:

$$\frac{2}{127} \to \frac{4}{127} \to \frac{8}{127} \to \frac{16}{127} \to \frac{32}{127} \to \frac{64}{127} \to \frac{126}{127} \to \frac{2}{127}$$

(Note that the first five numbers are less than $\frac{1}{2}$ so step A is applied to double the number; the sixth and seventh numbers are greater than $\frac{1}{2}$ so step B is applied.)

Suppose next that the chain is produced by applying steps AAAABBB in this order.
Then the chain is

$$x \to 2x \to 4x \to 8x \to 16x \to 2(1-16x) = 2-32x \to$$

$$2(1-(2-32x)) = 64x-2 \to 2(1-(64x-2)) = 6-128x$$

Since the eighth number equals the first, then $x = 6 - 128x$, and so $129x = 6$ from which $x = \dfrac{6}{129} = \dfrac{2}{43}$.
Thus, $m = 43$ is a possible value of m.
We double check this possible value by writing out the resulting chain:

$$\frac{2}{43} \to \frac{4}{43} \to \frac{8}{43} \to \frac{16}{43} \to \frac{32}{43} \to \frac{22}{43} \to \frac{42}{43} \to \frac{2}{43}$$

(Note that the first four numbers are less than $\frac{1}{2}$ so step A is applied to double the number; the fifth, sixth and seventh numbers are greater than $\frac{1}{2}$ so step B is applied.)

Therefore, three possible values of m are 43, 127 and 129.

(It turns out that these are the only three possible values of m with $m > 3$. This is a result of the fact that the eighth number in the chain will always equal $n + 128x$ or $n - 128x$ for some integer n. Can you see why the eighth number has to have this form, and how we can use this to prove that there are no other values of m?)