Canadian Intermediate Mathematics Contest

Tuesday, November 20, 2012
(in North America and South America)

Wednesday, November 21, 2012
(outside of North America and South America)

UNIVERSITY OF WATERLOO

WATERLOO MATHEMATICS

Time: 2 hours © 2012 University of Waterloo

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so.
There are two parts to this paper.

PART A

1. This part consists of six questions, each worth 5 marks.

2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.

2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.

3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

NOTES:
The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A. At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, http://www.cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

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PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. Jeanne is standing in a single line with several other people. There are four people in front of her and seven people behind her. How many people are in the line?

2. Which of the following is the largest: 1^{20}, 2^{14}, 4^8, 8^5, 16^3?

3. The length of a rectangle is three times its width. If the length is decreased by 5 and the width is increased by 5, the rectangle becomes a square. Determine the length of the original rectangle.

4. In the diagram, $\angle AFC = 90°$, D is on AC, $\angle EDC = 90°$, $CF = 21$, $AF = 20$, and $ED = 6$. Determine the total area of quadrilateral $AFCE$.
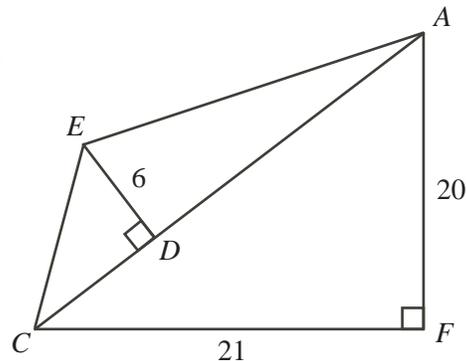
5. In the diagram, AB is a diameter of a circle with centre O. C and D are points on the circle. OD intersects AC at P, OC intersects BD at Q, and AC intersects BD at R. If $\angle BOQ = 60°$ and $\angle APO = 100°$, calculate the measure of $\angle BQO$.
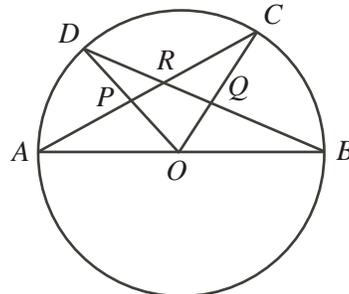
6. Integers can be written in bases other than the usual base 10. For example, the notation $(235)_7$ stands for the base 7 representation of the integer $2 \times 7^2 + 3 \times 7 + 5$ (which equals 124 in base 10). In general, if x, y and z are integers between 0 and $b - 1$, inclusive, then $(xyz)_b = xb^2 + yb + z$.

   Find all triples (x, y, z) for which $(xyz)_{10} = 2(xyz)_7$, where each of x, y and z comes from the list 1, 2, 3, 4, 5, 6.

PART B

For each question in Part B, your solution must be well organized and contain words of explanation or justification when appropriate. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. Rowan is given a square grid whose side length is an odd integer. The square is divided into 1 by 1 squares,. Rowan shades each 1 by 1 square along the two diagonals and does not shade any of the remaining squares. For example, a 5 by 5 grid is shown in which Rowan has shaded the squares along the diagonals.
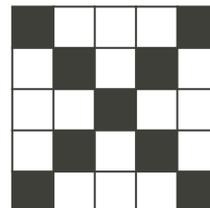
   (a) Rowan is given a square grid that is 7 by 7. How many 1 by 1 squares does he shade?

   (b) Rowan is given a square grid that is 101 by 101. Explain why the number of 1 by 1 squares that he shades is 201.

   (c) Rowan is given another square grid with odd side length, and the number of 1 by 1 squares that he shades is 41. How many *unshaded* 1 by 1 squares are there in the grid?

   (d) Rowan is given a fourth square grid with odd side length. After shading the squares on the diagonals, there are 196 *unshaded* 1 by 1 squares. How many 1 by 1 squares does this grid contain in total?

2. The line L_1 has equation $y = -\frac{4}{3}x$ and passes through the origin, O. The line L_2 has equation $y = -\frac{1}{2}x + 5$ and crosses the x-axis at P. Lines L_1 and L_2 intersect at Q.

   (a) What are the coordinates of points P and Q? (No justification is required.)

   (b) Find the area of $\triangle OPQ$.

   (c) Point R is on the positive x-axis so that the area of $\triangle OQR$ is three times the area of $\triangle OPQ$. Determine the coordinates of R.

   (d) Point S has coordinates $(18, t)$, with $t > 0$. If the area of $\triangle OQS$ is three times the area of $\triangle OPQ$, determine the value of t.

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3. Vernon starts with a first number n with $0 < n < 1$. Vernon enters the first number into a machine to produce a second number. He then enters the second number back into the machine to produce a third number. Vernon continually enters the result back into the machine giving a chain of numbers.

When Vernon enters a number x into the machine,

- if $x \le \frac{1}{2}$, the machine outputs $2x$, and
- if $x > \frac{1}{2}$, the machine outputs $2(1-x)$.

If the machine ever produces the number 1, Vernon stops the process.

(a) A chain starts with $\frac{3}{11}$. This gives $\frac{3}{11} \to \frac{6}{11} \to \frac{10}{11} \to \frac{2}{11} \to \cdots$. What are the next four numbers in this chain?

(b) Vernon enters a number x with $0 < x < 1$ into the machine and the machine produces the number x. Determine the value of x.

(c) The fourth number in a chain is 1. Determine all of the possible values of the first number in this chain.

(d) For some positive integers $m > 3$, if the first number in the chain is $\dfrac{2}{m}$, then the eighth number in the chain is also $\dfrac{2}{m}$. Determine three positive integers m for which this is true.