The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Grade 8 solutions follow the Grade 7 solutions

2011 Gauss Contests

(Grades 7 and 8)

Wednesday, May 11, 2011

Solutions

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Grade 7

1. Evaluating, $5+4-3+2-1=9-3+2-1=6+2-1=8-1=7$.

   Answer: (E)

2. We must first add 9 and 16. Thus, $\sqrt{9+16}=\sqrt{25}=5$.

   Answer: (E)

3. Reading from the bar graph, only 1 student chose spring.
   Since 10 students were surveyed, then the percentage of students that chose spring was $\frac{1}{10}\times 100\%$ or 10%.

   Answer: (B)

4. Since ground beef sells for \$5.00 per kg, then the cost of 12 kg is $\$5.00\times 12=\$60.00$.

   Answer: (C)

5. Since each of the numbers is between 1 and 2, we consider the tenths digits.
   The numbers 1.0101, 1.0011 and 1.0110 are between 1 and 1.1, while 1.1001 and 1.1100 are both greater than 1.1.
   We may eliminate answers (D) and (E).
   Next, we consider the hundredths digits.
   While 1.0101 and 1.0110 each have a 1 as their hundreths digit, 1.0011 has a 0 and is therefore the smallest number in the list.
   The ordered list from smallest to largest is $\{1.0011, 1.0101, 1.0110, 1.1001, 1.1100\}$

   Answer: (B)

6. Since you *randomly* choose one of the five answers, then each has an equally likely chance of being selected.
   Thus, the probability that you select the one correct answer from the five is $\frac{1}{5}$.

   Answer: (A)

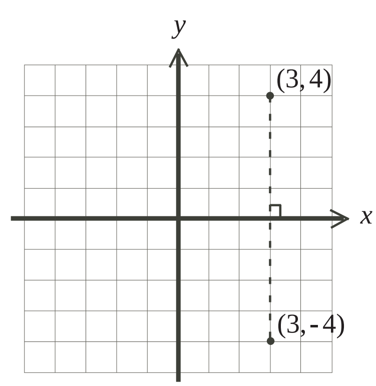
7. Since we are adding $\frac{1}{3}$ seven times, then the result is equal to $7\times\frac{1}{3}$.

   Answer: (E)

8. Keegan paddled 12 km of his 36 km trip before lunch.
   Therefore, Keegan has $36-12=24$ km left to be completed after lunch.
   The fraction of his trip remaining to be completed is $\frac{24}{36}=\frac{2}{3}$.

   Answer: (D)

9. After reflecting the point $(3,4)$ in the x-axis, the x-coordinate of the image will be the same as the x-coordinate of the original point, $x=3$.
   The original point is a distance of 4 from the x-axis.
   The image will be the same distance from the x-axis, but below the x-axis.
   Thus, the image has y-coordinate -4.
   The coordinates of the image point are $(3,-4)$.



   Answer: (D)

10. Anika said that the plant was a *red* rose.
Cathy said that the plant was a *red* dahlia.
If red was not the correct colour of the plant, then Anika was incorrect about the colour and therefore must be correct about the type; in other words, the plant is a rose.
Similarly, if red was not the correct colour of the plant, then Cathy was incorrect about the colour and therefore must be correct about the type; in other words, the plant is a dahlia.
But the plant cannot be both a rose and a dahlia.
Therefore, Cathy and Anika must have been correct about the colour being red.
Bill said that the plant was a purple daisy.
Since we know that the colour of the plant is red, then Bill was incorrect about it being purple.
Therefore, Bill must have been correct about it being a daisy.
Thus, the plant is a red daisy.

ANSWER: (E)

11. The angles $2x^\circ$ and $3x^\circ$ shown are complementary and thus add to 90°.
Therefore, $2x + 3x = 90$ or $5x = 90$ and so $x = \frac{90}{5} = 18$.

ANSWER: (D)

12. Since the four sides of a square are equal in length and the perimeter is 28, then each side has length $\frac{28}{4} = 7$.
The area of the square is the product of the length and width, which are each equal to 7.
Therefore, the area of the square in cm^2 is $7 \times 7 = 49$.

ANSWER: (D)

13. Since Kayla ate less than Max and Chris ate more than Max, then Kayla ate less than Max who ate less than Chris.
Brandon and Tanya both ate less than Kayla.
Therefore, Max ate the second most.

ANSWER: (D)

14. The smallest three digit palindrome is 101.
The largest three digit palindrome is 999.
The difference between the smallest three digit palindrome and the largest three digit palindrome is $999 - 101 = 898$.

ANSWER: (B)

15. Since 10 minutes is equivalent to $\frac{10}{60} = \frac{1}{6}$ of an hour, the skier travels $12 \div 6 = 2$ km.

ANSWER: (C)

16. Any number of 2 cm rods add to give a rod having an even length.
Since we need an odd length, 51 cm, then we must combine an odd length from the 5 cm rods with the even length from the 2 cm rods to achieve this.
An odd length using 5 cm rods can only be obtained by taking an odd number of them.
All possible combinations are shown in the table below.

Number of 5 cm rods	Length in 5 cm rods	Length in 2 cm rods	Number of 2 cm rods
1	5	$51 - 5 = 46$	$46 \div 2 = 23$
3	15	$51 - 15 = 36$	$36 \div 2 = 18$
5	25	$51 - 25 = 26$	$26 \div 2 = 13$
7	35	$51 - 35 = 16$	$16 \div 2 = 8$
9	45	$51 - 45 = 6$	$6 \div 2 = 3$

Note that attempting to use 11 (or more) 5 cm rods gives more than the 51 cm length required.
Thus, there are exactly 5 possible combinations that add to 51 cm using 5 cm rods first followed by 2 cm rods.

ANSWER: (A)

17. *Solution 1*
Choosing one meat and one fruit, the possible lunches are beef and apple, beef and pear, beef and banana, chicken and apple, chicken and pear, or chicken and banana.
Of these 6 lunches, 2 of them include a banana.
Thus when randomly given a lunch, the probability that it will include a banana is $\frac{2}{6}$ or $\frac{1}{3}$.

Solution 2
Each of the possible lunches that Braydon may receive contain exactly one fruit.
The meat chosen for each lunch does not affect what fruit is chosen.
Thus, the probability that the lunch includes a banana is independent of the meat that is served with it.
Since there are 3 fruits to choose from, then the probability that the lunch includes a banana is $\frac{1}{3}$.

ANSWER: (A)

18. *Solution 1*
In kilograms, let the weights of the 3 pumpkins in increasing order be A, B and C.
The lightest combined weight, 12 kg, must come from weighing the two lightest pumpkins.
That is, $A + B = 12$.
The heaviest combined weight, 15 kg, must come from weighing the two heaviest pumpkins.
That is, $B + C = 15$.
Then the third given weight, 13 kg, is the combined weight of the lightest and heaviest pumpkins.
That is, $A + C = 13$.
Systematically, we may try each of the 5 possible answers.
If the lightest pumpkin weighs 4 kg (answer (A)), then $A + B = 12$ gives $4 + B = 12$, or $B = 8$.
If $B = 8$, then $8 + C = 15$ or $C = 7$.
Since C represents the weight of the heaviest pumpkin, C cannot be less than B and therefore the lightest pumpkin cannot weigh 4 kg.
If the lightest pumpkin weighs 5 kg (answer (B)), then $A + B = 12$ gives $5 + B = 12$, or $B = 7$.
If $B = 7$, then $7 + C = 15$ or $C = 8$.
If $A = 5$ and $C = 8$, then the third and final equation $A + C = 13$ is also true.
Therefore, the weight of the smallest pumpkin must be 5 kg.

Solution 2
In kilograms, let the weights of the 3 pumpkins in increasing order be A, B and C.
The lightest combined weight, 12 kg, must come from weighing the two lightest pumpkins.
That is, $A + B = 12$.
The heaviest combined weight, 15 kg, must come from weighing the two heaviest pumpkins.
That is, $B + C = 15$.
Then the third given weight, 13 kg, is the combined weight of the lightest and heaviest pumpkins.
That is, $A + C = 13$.
Since $A + B = 12$ and $A + C = 13$, then C is one more than B.
Since $B + C = 15$ and C is one more than B, then $B = 7$ and $C = 8$.

Since $A + B = 12$, then $A + 7 = 12$ or $A = 5$.
When $A = 5$, $B = 7$ and $C = 8$, the weights of the pairs of pumpkins are 12 kg, 13 kg and 15 kg as was given.
Therefore, the weight of the lightest pumpkin is 5 kg.

ANSWER: (B)

19. If each of the four numbers is increased by 1, then the increase in their sum is 4.
That is, these four new numbers when added together have a sum that is 4 more than their previous sum T, or $T + 4$.
This new sum $T + 4$ is now tripled.
The result is $3 \times (T + 4) = (T + 4) + (T + 4) + (T + 4)$ or $3T + 12$.

ANSWER: (C)

20. The volume of the rectangular prism equals the area of its base 6×4 times its height 2.
That is, the rectangular prism has a volume of $6 \times 4 \times 2 = 48$ cm^3.
The volume of the triangular prism is found by multiplying the area of one of its triangular faces by its length.
The triangular face has a base of length $6 - 3 = 3$ cm.
This same triangular face has a perpendicular height of $5 - 2 = 3$ cm, since the height of the rectangular prism is 2 cm.
Thus, the triangular face has area $\frac{3\times 3}{2} = \frac{9}{2}$ cm^2.
Since the length of the triangular prism is 4 cm, then its volume is $\frac{9}{2} \times 4 = \frac{36}{2} = 18$ cm^3.
The volume of the combined structure is equal to the sum of the volumes of the two prisms, or $48 + 18 = 66$ cm^3.

ANSWER: (E)

21. Steve counts forward by 3 beginning at 7.
That is, the numbers that Steve counts are each 7 more than some multiple of 3.
We can check the given answers to see if they satisfy this requirement by subtracting 7 from each of them and then determining if the resulting number is divisible by 3.
We summarize the results in the table below.

Answers	Result after subtracting 7	Divisible by 3?
1009	1002	Yes
1006	999	Yes
1003	996	Yes
1001	994	No
1011	1004	No

Of the possible answers, Steve only counted 1009, 1006 and 1003.
Dave counts backward by 5 beginning at 2011.
That is, the numbers that Dave counts are each some multiple of 5 less than 2011.
We can check the given answers to see if they satisfy this requirement by subtracting each of them from 2011 and then determining if the resulting number is divisible by 5.
We summarize the results in the table below.

Answers	Result after being subtracted from 2011	Divisible by 5?
1009	1002	No
1006	1005	Yes
1003	1008	No
1001	1010	Yes
1011	1000	Yes

Of the possible answers, Dave only counted 1006, 1001 and 1011.
Thus while counting, the only answer that both Steve and Dave will list is 1006.

Answer: (B)

22. In the first 20 minutes, Sheila fills the pool at a rate of 20 L/min and thus adds $20 \times 20 = 400$ L of water to the pool.
At this time, the pool needs $4000 - 400 = 3600$ L of water to be full.
After filling for 20 minutes, water begins to leak out of the pool at a rate of 2 L/min.
Since water is still entering the pool at a rate of 20 L/min, then the net result is that the pool is filling at a rate of $20 - 2 = 18$ L/min.
Since the pool needs 3600 L of water to be full and is filling at a rate of 18 L/min, then it will take an additional $3600 \div 18 = 200$ minutes before the pool is full of water.
Thus, the total time needed to fill the pool is $20 + 200 = 220$ minutes or 3 hours and 40 minutes.

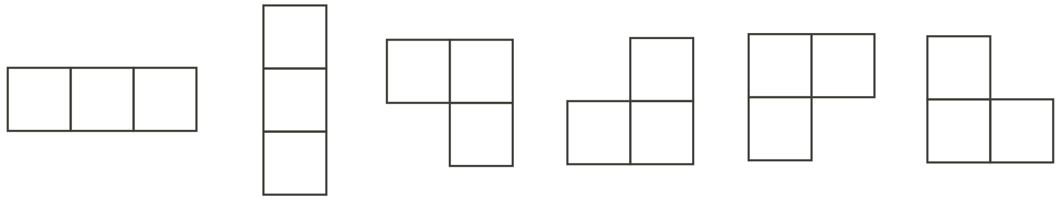
Answer: (B)

23. The sum of the units column is $E + E + E = 3E$.
Since E is a single digit, and $3E$ ends in a 1, then the only possibility is $E = 7$.
Then $3E = 3 \times 7 = 21$, and thus 2 is carried to the tens column.
The sum of the tens column becomes $2 + B + C + D$.
The sum of the hundreds column is $A + A + A = 3A$ plus any carry from the tens column.
Thus, $3A$ plus the carry from the tens column is equal to 20.
If there is no carry from the tens column, then $3A = 20$.
This is not possible since A is a single digit positive integer.
If the carry from the tens column is 1, then $3A + 1 = 20$ or $3A = 19$.
Again, this is not possible since A is a single digit positive integer.
If the carry from the tens column is 2, then $3A + 2 = 20$ or $3A = 18$ and $A = 6$.
Since B, C, and D are single digits (ie. they are each less than or equal to 9), then it is not possible for the carry from the tens column to be greater than 2.
Therefore, $A = 6$ is the only possibility.
Since the carry from the tens column is 2, then the sum of the tens column, $2 + B + C + D$, must equal 21.
Thus, $2 + B + C + D = 21$ or $B + C + D = 19$.
Since $A = 6$ and $E = 7$, then the sum $A + B + C + D + E = 6 + 19 + 7 = 32$.
Note that although we don't know B, C and D, it is only necessary that their sum be 19.
Although there are many possibilities, the example with $B = 2$, $C = 8$ and $D = 9$ is shown.

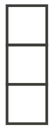
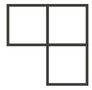
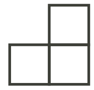
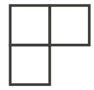
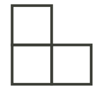
$$\begin{array}{r} 6\ 2\ 7 \\ 6\ 8\ 7 \\ +\ 6\ 9\ 7 \\ \hline 2\ 0\ 1\ 1 \end{array}$$

Answer: (C)

24. First we recognize that given the conditions for the three selected squares, there are only 6 possible shapes that may be chosen. These are shown below.



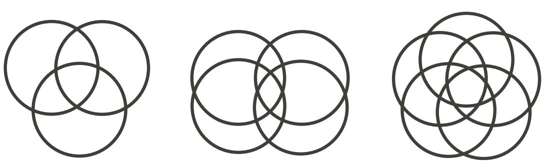
To determine the number of ways that three squares can be selected, we count the number of ways in which each of these 6 shapes can be chosen from the given figure.
The results are summarized in the table below.

Shape						
Number of Each	3	2	4	3	3	4

Thus, three of the nine squares can be selected as described in $3+2+4+3+3+4=19$ ways.

Answer: (A)

25. We first note that each circle can intersect any other circle a maximum of two times.
To begin, the first circle is drawn.
The second circle is then drawn overlapping the first, and two points of intersection are created.
Since each pair of circles overlap (but are not exactly on top of one another), then the third circle drawn can intersect the first circle twice and the second circle twice.
We continue in this manner with each new circle drawn intersecting each of the previously drawn circles exactly twice.
That is, the third circle drawn intersects each of the two previous circles twice, the fourth circle intersects each of the three previous circles twice, and so on.
Diagrams showing possible arrangements for 3, 4 and 5 circles, each giving the maximum number of intersections, are shown below.



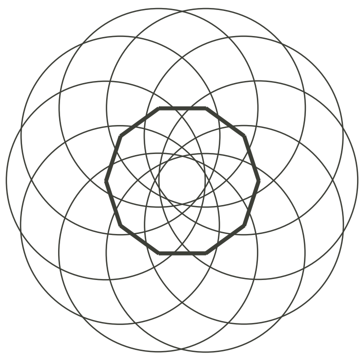
The resulting numbers of intersections are summarized in the table below.

Circle number drawn	Number of new intersections	Total number of intersections
1	0	0
2	2	2
3	$2\times 2=4$	$2+4$
4	$3\times 2=6$	$2+4+6$
5	$4\times 2=8$	$2+4+6+8$
6	$5\times 2=10$	$2+4+6+8+10$
7	$6\times 2=12$	$2+4+6+8+10+12$
8	$7\times 2=14$	$2+4+6+8+10+12+14$
9	$8\times 2=16$	$2+4+6+8+10+12+14+16$
10	$9\times 2=18$	$2+4+6+8+10+12+14+16+18$

Thus, the greatest possible total number of intersection points using ten circles is

$$2+4+6+8+10+12+14+16+18=90.$$

To be complete, we technically need to show that this number is possible, though we don't expect students to do this to answer the question.
The diagram below demonstrates a possible positioning of the ten circles that achieves the maximum 90 points of intersection.
That is, every pair of circles intersects exactly twice and all points of intersection are distinct from one another.
It is interesting to note that this diagram is constructed by positioning each of the ten circles' centres at one of the ten vertices of a suitably sized regular decagon, as shown.



Answer: (D)

Grade 8

1. The fractions $\frac{8}{12}$ and $\frac{\square}{3}$ are equivalent fractions.
   To reduce the first to the second, the denominator 12 has been divided by a factor of 4. Therefore, we divide the numerator 8 by the same factor, 4.
   Thus, $\frac{8}{12} = \frac{2}{3}$ and the value represented by $\square$ is 2.

   Answer: (D)

2. Since ground beef sells for \$5.00 per kg, then the cost of 12 kg is $\$5.00 \times 12 = \$60.00$.

   Answer: (C)

3. When the unknown angle y° is added to the 90° angle, the result is a complete rotation, or 360°.
   Thus, $y^\circ + 90^\circ = 360^\circ$ or $y = 360 - 90 = 270$.

   Answer: (E)

4. *Solution 1*
   To compare the five fractions we rewrite them each with a common denominator of 100.
   Then the list equivalent to the given list $\left\{\frac{3}{10}, \frac{9}{20}, \frac{12}{25}, \frac{27}{50}, \frac{49}{100}\right\}$ is $\left\{\frac{30}{100}, \frac{45}{100}, \frac{48}{100}, \frac{54}{100}, \frac{49}{100}\right\}$.
   The largest number in this new list is $\frac{54}{100}$, and therefore $\frac{27}{50}$ is the largest number in the given list.

   Solution 2
   We notice that with the exception of $\frac{27}{50}$, each numerator in the given list is less than one half of its corresponding denominator.
   Thus, each fraction except $\frac{27}{50}$ has a value that is less than one half.
   The fraction $\frac{27}{50}$ is larger than one half.
   Therefore $\frac{27}{50}$ is the largest number in the list.

   Answer: (D)

5. Since 3 of the 15 balls are red, then the probability that Alex randomly selects a red ball is $\frac{3}{15}$ or $\frac{1}{5}$.

   Answer: (A)

6. *Solution 1*
   Since double the original number plus 3 is 23, then double the original number must equal 20 (that is, $23 - 3$).
   Therefore, the original number is 20 divided by 2, or 10.

   Solution 2
   Let the original number be represented by the variable x.
   Then doubling the original number and adding 3 gives $2x + 3$.
   Thus, $2x + 3 = 23$ or $2x = 23 - 3 = 20$, so $x = \frac{20}{2} = 10$.

   Answer: (B)

7. *Solution 1*
   To make half of the recipe, only half of the $4\frac{1}{2}$ cups of flour are needed.
   Since half of 4 is 2 and half of $\frac{1}{2}$ is $\frac{1}{4}$, then $2\frac{1}{4}$ cups of flour are needed.

   Solution 2
   To make half of the recipe, only half of the $4\frac{1}{2}$ cups of flour are needed.
   To determine half of $4\frac{1}{2}$, we divide $4\frac{1}{2}$ by 2, or multiply by $\frac{1}{2}$.
   Thus, $4\frac{1}{2} \times \frac{1}{2} = \frac{9}{2} \times \frac{1}{2} = \frac{9}{4} = 2\frac{1}{4}$ cups of flour are needed to make half of the recipe.

   Answer: (B)

8. Since $\angle PQR = \angle PRQ$, then $\triangle PQR$ is an isosceles triangle and $PQ = PR = 7$.
Therefore, the perimeter of $\triangle PQR$ is $PQ + QR + PR = 7 + 5 + 7 = 19$.

ANSWER: (E)

9. If 15 of 27 students in the class are girls, then the remaining $27 - 15 = 12$ students are boys.
The ratio of boys to girls in the class is $12 : 15 = 4 : 5$.

ANSWER: (A)

10. Since Kayla ate less than Max and Chris ate more than Max, then Kayla ate less than Max who ate less than Chris.
Brandon and Tanya both ate less than Kayla.
Therefore, Max ate the second most.

ANSWER: (D)

11. Evaluating each of the expressions,
(A): $(2 \times 3)^2 = 6^2 = 36$
(B): $3 + 2^2 = 3 + 4 = 7$
(C): $2^3 - 1 = 8 - 1 = 7$
(D): $3^2 - 2^2 = 9 - 4 = 5$
(E): $(3 + 2)^2 = 5^2 = 25$,
we see that only expression (D) is equal to 5.

ANSWER: (D)

12. Nick charges \$10 per hour of babysitting.
If Nick babysits for y hours, then his charge just for babysitting is $10y$ dollars.
In addition, Nick charges a one-time fee of \$7 for travel costs.
Thus, the expression that represents the total number of dollars that Nick charges for y hours of babysitting is $10y + 7$.

ANSWER: (A)

13. Measured in cm^2, the area of Kalob's window is 50×80.
Measured in cm^2, twice the area of Kalob's window is $50 \times 80 \times 2$ which is equal to 50×160.
Thus, a window with dimensions 50 cm $\times$ 160 cm is a window with area double the area of Kalob's window.

ANSWER: (C)

14. First recognize that the day and the month must be equal.
Next, since $3^2 = 9$ and $10^2 = 100$, both the day and the month must be larger than 3 but less than 10 so that the year lies between 2012 and 2099.
We list all possible square root days in the table below.

Day and Month	Last Two Digits of the Year	Date
4	$4^2 = 16$	4/4/2016
5	$5^2 = 25$	5/5/2025
6	$6^2 = 36$	6/6/2036
7	$7^2 = 49$	7/7/2049
8	$8^2 = 64$	8/8/2064
9	$9^2 = 81$	9/9/2081

Since these are all actual dates between January 1, 2012 and December 31, 2099, then the number of square root days is 6.

ANSWER: (E)

15. In $\triangle CDE$, $CE = 5$, $DE = 3$, and $\angle CDE = 90°$.
By the Pythagorean Theorem, $CE^2 = CD^2 + DE^2$ or $CD^2 = CE^2 - DE^2$ or
$CD^2 = 5^2 - 3^2 = 25 - 9 = 16$, so $CD = 4$ (since $CD > 0$).
In $\triangle ABC$, $AB = 9$, $BC = BD - CD = 16 - 4 = 12$, and $\angle ABC = 90°$.
By the Pythagorean Theorem, $AC^2 = AB^2 + BC^2$ or $AC^2 = 9^2 + 12^2 = 81 + 144 = 225$, so $AC = 15$ (since $AC > 0$).

Answer: (C)

16. *Solution 1*
Beatrix is twice the height of Violet who is $\frac{2}{3}$ the height of Georgia.
Therefore, Beatrix is 2 times $\frac{2}{3}$ the height of Georgia, or $\frac{4}{3}$ the height of Georgia.

Solution 2
Let the heights of Beatrix, Violet and Georgia be represented by B, V and G respectively.
Since Beatrix is twice the height of Violet, then $B = 2V$.
Since Violet is $\frac{2}{3}$ the height of Georgia, then $V = \frac{2}{3}G$.
Substituting $\frac{2}{3}G$ for V in the first equation, we get $B = 2V = 2(\frac{2}{3}G) = \frac{4}{3}G$.
Thus, Beatrix's height is $\frac{4}{3}$ of Georgia's height.

Answer: (C)

17. Since x can be any value in between 0 and 1, we choose a specific value for x in this range.
For example, we allow x to be $\frac{1}{4}$ and then evaluate each of the five expressions.
We get, $x = \frac{1}{4}$; $x^2 = (\frac{1}{4})^2 = \frac{1}{16}$; $2x = 2\left(\frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}$; $\sqrt{x} = \sqrt{\frac{1}{4}} = \frac{1}{2}$; $\frac{1}{x} = \frac{1}{\frac{1}{4}} = 4$.
Since $\frac{1}{16}$ is the smallest value, then for any value of x between 0 and 1, x^2 will produce the smallest value of the five given expressions.
In fact, no matter what x between 0 and 1 is chosen, x^2 is always smallest.

Answer: (B)

18. Assume that each square has side length 2, and thus has area 4.
In square $ABCD$, diagonal AC divides the square into 2 equal areas.
Thus, the area of $\triangle ACD$ is one half of the area of square $ABCD$ or 2.
Since AC is the diagonal of square $ABCD$, $\angle ACD = \angle ACB = 90° \div 2 = 45°$.
Also, $\angle DCH = \angle BCH - \angle DCB = 180° - 90° = 90°$.
Since ACJ is a straight line segment, $\angle ACJ = \angle ACD + \angle DCH + \angle HCJ = 180°$.
Thus, $\angle HCJ = 180° - \angle ACD - \angle DCH = 180° - 45° - 90° = 45°$.
In $\triangle CHJ$, $\angle HCJ = 45°$ and $\angle CHJ = 90°$, so $\angle HJC = 180° - 90° - 45 = 45°$.
Therefore $\triangle CHJ$ is isosceles, so $CH = HJ = 1$, since J is the midpoint of GH.
Therefore $\triangle CHJ$ has area $\frac{1}{2} \times 1 \times 1$ or $\frac{1}{2}$.
Since $\triangle ACD$ has area 2, and $\triangle CHJ$ has area $\frac{1}{2}$, then the combined areas of the two shaded regions is $2 + \frac{1}{2}$ or $\frac{5}{2}$.
Since the total area of the two large squares is 8, the fraction of the two squares that is shaded is $\frac{5}{2} \div 8 = \frac{5}{2} \times \frac{1}{8} = \frac{5}{16}$.

Answer: (D)

19. We must consider that the integers created could be one-digit, two-digit or three-digit integers.
First, consider one-digit integers.
Since 1, 2 and 3 are the only digits that may be used, then there are only 3 one-digit positive integers less than 400, namely 1, 2 and 3.
Next, consider the number of two-digit integers that can be created.
Since digits may be repeated, the integers 11, 12, 13, 21, 22, 23, 31, 32, 33 are the only possibilities.

Thus, there are 9 two-digit positive integers that can be created.
Instead of listing these 9 integers, another way to count how many there are is to consider that there are 3 choices for the first digit (either 1, 2 or 3 may be used) and also 3 choices for the second digit (since repetition of digits is allowed), so $3 \times 3 = 9$ possibilities.
Finally, we count the number of three-digit positive integers.
There are 3 choices for the first digit, 3 choices for the second digit and 3 choices for the third digit.
Thus, there are $3 \times 3 \times 3 = 27$ possible three-digit positive integers that can be created. (Note that all of these are less than 400, since the largest of them is 333.)
In total, the number of positive integers less than 400 that can be created using only the digits 1, 2 or 3 (with repetition allowed), is $3 + 9 + 27 = 39$.

Answer: (D)

20. Since the average height of all 22 students is 103 cm, then the sum of the heights of all students in the class is $22 \times 103 = 2266$ cm.
Since the average height of the 12 boys in the class is 108 cm, then the sum of the heights of all boys in the class is $12 \times 108 = 1296$ cm.
The sum of the heights of all the girls in class is the combined height of all students in the class less the height of all the boys in the class, or $2266 - 1296 = 970$ cm.
Since there are 10 girls in the class, their average height is $970 \div 10 = 97$ cm.

Answer: (B)

21. We first consider the minimum number of coins needed to create 99¢, the greatest amount that we are required to create.
We can create 75¢ with a minimum number of coins by placing three quarters in the collection.
Once at 75¢, we need $99 - 75 = 24$¢ more to reach 99¢.
To create 24¢ with a minimum number of coins we use two dimes and 4 pennies.
Therefore, the minimum number of coins required to create 99¢ is 3 quarters, 2 dimes and 4 pennies, or 9 total coins in the collection.
In fact, this is the only group of 9 coins that gives exactly 99¢.
To see this, consider that we must have the 4 pennies and the 3 quarters, but these 7 coins only give 79¢.
We now need exactly 20¢ and have only two coins left to include in our collection.
Therefore the two remaining coins must both be dimes, and thus the only group of 9 coins that gives exactly 99¢ is 3 quarters, 2 dimes and 4 pennies.
We now attempt to check if all other amounts of money less than one dollar can be created using only these 9 coins.
Using the 4 pennies, we can create each of the amounts 1¢, 2¢, 3¢, and 4¢.
However, we don't have a way to create 5¢ using 3 quarters, 2 dimes and 4 pennies.
Thus, at least one more coin is needed.
It is clear that if the 10th coin that we add to the collection is a penny, then we will have 5 pennies and thus be able to create 5¢.
However, with 3 quarters, 2 dimes and 5 pennies, we won't be able to create 9¢.
If instead of a penny we add a nickel as our 10th coin in the collection, then we will have 3 quarters, 2 dimes, 1 nickel and 4 pennies.
Obviously then we can create 5¢ using the 1 nickel.
In fact, we are now able to create the following sums of money:

- any amount from 1¢ to 4¢ using 4 pennies;

- any amount from 5¢ to 9¢ using 1 nickel and 4 pennies;
- any amount from 10¢ to 14¢ using 1 dime and 4 pennies;
- any amount from 15¢ to 19¢ using 1 dime, 1 nickel and 4 pennies;
- any amount from 20¢ to 24¢ using 2 dimes and 4 pennies.

If in addition to the amounts from 1¢ to 24¢ we use a quarter, we will be able to create any amount from 25¢ to 49¢.
Similarly, using 2 quarters we can create any amount from 50¢ to 74¢.
Finally, using 3 quarters we can create any amount from 75¢ to 99¢.
Therefore, the smallest number of coins needed to create any amount of money less than one dollar is 10.

Can you verify that there is a different combination of 10 coins with which we can create any amount of money less than one dollar?

Answer: (A)

22. Consider the possible ways that the numbers from 1 to 9 can be used three at a time to sum to 18.
Since we may not repeat any digit in the sum, the possibilities are:
$1+8+9, 2+7+9, 3+6+9, 3+7+8, 4+5+9, 4+6+8, 5+6+7$.
Next, consider the row $1+d+f$ in Fig.1.
Since the sum of every row is 18, $d+f=17$.
This gives that either $d=8$ and $f=9$ or $d=9$ and $f=8$.
Next, consider the 3 rows in which x appears.
The sums of these 3 rows are $a+x+d$, $b+x+f$, and $c+x+6$.
That is, x appears in exactly 3 distinct sums.
Searching our list of possible sums above, we observe that only the numbers $6, 7$ and 8 appear in 3 distinct sums.
That is, x must equal either $6, 7$ or 8.
However, the number 6 already appears in the table.
Thus, x is not 6.
Similarly, we already concluded that either d or f must equal 8.
Thus, x is not 8. Therefore, $x=7$ is the only possibility.
Fig.2 shows the completed table and verifies that the number represented by x is indeed 7.

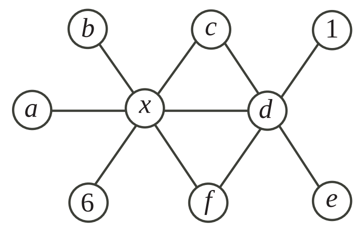


Fig.1

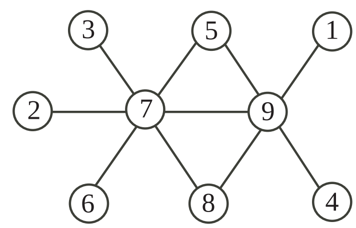
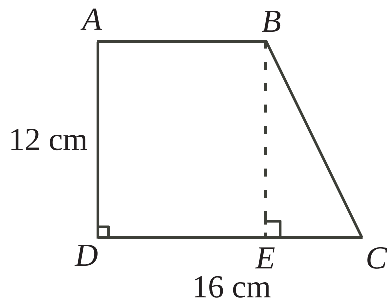


Fig.2

Answer: (C)

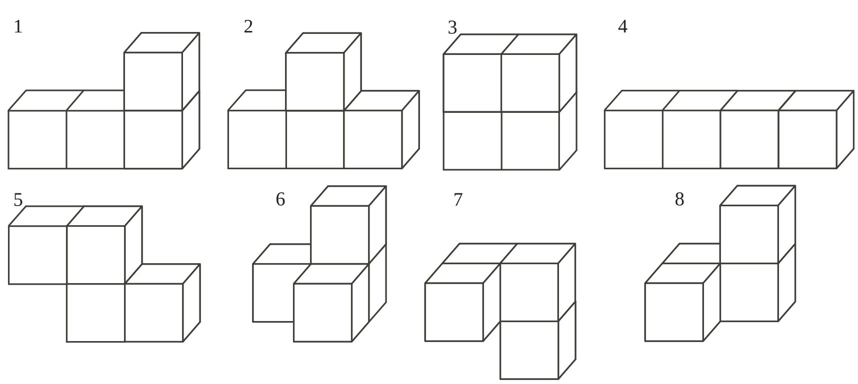
23. We first label the trapezoid $ABCD$ as shown in the diagram below.
Since AD is the perpendicular height of the trapezoid, then AB and DC are parallel.
The area of the trapezoid is $\frac{AD}{2} \times (AB + DC)$ or $\frac{12}{2} \times (AB + 16)$ or $6 \times (AB + 16)$.
Since the area of the trapezoid is 162, then $6 \times (AB + 16) = 162$ and $AB + 16 = \frac{162}{6}$ or $AB + 16 = 27$, so $AB = 11$.
Construct a perpendicular from B to E on DC.
Since AB is parallel to DE and both AD and BE are perpendicular to DE, then $ABED$ is a rectangle.
Thus, $DE = AB = 11$, $BE = AD = 12$, and $EC = DC - DE = 16 - 11 = 5$.
Since $\angle BEC = 90°$, then $\triangle BEC$ is a right-angled triangle.
Thus by the Pythagorean Theorem, $BC^2 = BE^2 + EC^2$ or $BC^2 = 12^2 + 5^2$ or $BC^2 = 169$ so

$BC = 13$ (since $BC > 0$).
The perimeter of the trapezoid is $AB + BC + CD + DA = 11 + 13 + 16 + 12 = 52$ cm.



ANSWER: (B)

24. When Ada glues cube faces together, they must coincide.
Also, each of the 4 cubes must have a face that coincides with a face of at least one of the other 3 cubes.
We this in mind, we begin by repositioning the 4 identical cubes, attempting to construct figures different from those already constructed.
Figure 1 shown below was given in the question.
The next 4 figures (numbered 2 to 5) have the same thickness as figure 1.
That is, each of the first 5 figures is 1 cube thick from front to back.
These are the only 5 figures that can be constructed having this 1 cube thickness and these 5 figures are all unique.
Can you verify this for yourself? Attempt to construct a different figure with a 1 cube thickness.
Now rotate this figure to see if it will match one of the first 5 figures shown.
Next, we consider constructing figures with a front to back depth of 2 cubes.
The only 3 unique figures that can be constructed in this way are labeled 6 to 8 below.
While figures 7 and 8 look to be the same, there is no way to rotate one of them so that it is identical to the other.
Thus, not only are these 3 figures the only figures that can be constructed with a 2 cube thickness, but they are all different from one another.
Since no rotation of any of these 3 figures will result in it having a 1 cube thickness from front to back, the figures 6 to 8 are all different than any of the previous 5 figures.
With some rotation, we can verify that the only figures having a 3 cube thickness have already been constructed (figures 1, 2 and 5).
Similarly, the only figure that can possibly be built with a 4 cube thickness has already been constructed (figure 4).
The 8 figures shown below are the only figures that Ada can construct.



ANSWER: (D)

25. The given sequence allows for many different patterns to be discovered depending on how terms in the sequence are grouped and then added.
One possibility is to add groups of four consecutive terms in the sequence.
That is, consider finding the sum of the sequence, S, in the manner shown below.

$$S = (1 + (-4) + (-9) + 16) + (25 + (-36) + (-49) + 64) + (81 + (-100) + (-121) + 144) + \ldots$$

The pattern that appears when grouping terms in this way is that each consecutive group of 4 terms, beginning at the first term, adds to 4.
That is, $1+(-4)+(-9)+16 = 4$, $25+(-36)+(-49)+64 = 4$, $81+(-100)+(-121)+144 = 4$, and so on.
For now, we will assume that this pattern of four consecutive terms adding to 4 continues and wait to verify this at the end of the solution.
Since each consecutive group of four terms adds to 4, the first eight terms add to 8, the first twelve terms add to 12 and the first n terms add to n provided that n is a multiple of 4.
Thus, the sum of the first 2012 terms is 2012, since 2012 is a multiple of 4.
Since we are required to find the sum of the first 2011 terms, we must subtract the value of the 2012^{th} term from our total of 2012.
We know that the n^{th} term in the sequence is either n^2 or it is $-n^2$.
Therefore, we must determine if the 2012^{th} term is positive or negative.
By the alternating pattern of the signs, the first and fourth terms in each of the consecutive groupings will be positive, while the second and third terms are negative.
Since the 2012^{th} term is fourth in its group of four, its sign is positive.
Thus, the 2012^{th} term is 2012^2.
Therefore, the sum of the first 2011 terms is the sum of the first 2012 terms, which is 2012, less the 2012^{th} term, which is 2012^2.
Thus, $S = 2012 - 2012^2 = 2012 - 4048144 = -4046132$.

Verifying the Pattern

While we do not expect that students will verify this pattern in the context of a multiple choice contest, it is always important to verify patterns.
One way to verify that the sum of each group of four consecutive terms (beginning with the first term) adds to 4, is to use algebra.
If the first of four consecutive integers is n, then the next three integers in order are $n + 1$, $n + 2$, and $n + 3$.
Since the terms in our sequence are the squares of consecutive integers, we let n^2 represent the first term in the group of four.
The square of the next integer larger than n is $(n + 1)^2$, and thus the remaining two terms in the group are $(n + 2)^2$ and $(n + 3)^2$.
Since the first and fourth terms are positive, while the second and third terms are negative, the sum of the four terms is $n^2 - (n + 1)^2 - (n + 2)^2 + (n + 3)^2$.
(For example, if $n = 5$ then we have $5^2 - 6^2 - 7^2 + 8^2$.)
To simplify this expression, we must first understand how to simplify its individual parts such as $(n + 1)^2$.
The expression $(n + 1)^2$ means $(n + 1) \times (n + 1)$.
To simplify this product, we multiply the n in the first set of brackets by each term in the second set of brackets and do the same for the 1 appearing in the first set of brackets.
The operation between each product remains as it appears in the expression, as an addition.

That is,

$$\begin{aligned}(n+1)^2 &= (n+1)\times(n+1)\\ &= n\times n + n\times 1 + 1\times n + 1\times 1\\ &= n^2+n+n+1\\ &= n^2+2n+1\end{aligned}$$

Applying this process again,
$(n+2)^2 = (n+2)\times(n+2) = n\times n + n\times 2 + 2\times n + 2\times 2 = n^2+2n+2n+4 = n^2+4n+4$,
and
$(n+3)^2 = (n+3)\times(n+3) = n\times n + n\times 3 + 3\times n + 3\times 3 = n^2+3n+3n+9 = n^2+6n+9$.
Therefore, the sum of each group of four consecutive terms (beginning with the first term) is,

$$\begin{aligned}n^2-(n+1)^2-(n+2)^2+(n+3)^2 &= n^2-(n^2+2n+1)-(n^2+4n+4)+(n^2+6n+9)\\ &= n^2-n^2-2n-1-n^2-4n-4+n^2+6n+9\\ &= n^2-n^2-n^2+n^2-2n-4n+6n+9-1-4\\ &= 4\end{aligned}$$

Answer: (E)