The CENTRE for EDUCATION in
MATHEMATICS and COMPUTING

2011 Galois Contest

Wednesday, April 13, 2011

Solutions

1. (a) Using Jackson's rule, the second term of Fabien's sequence is $\dfrac{1}{1-2} = \dfrac{1}{-1} = -1$.

   (b) Since the second term is -1, the third term is $\dfrac{1}{1-(-1)} = \dfrac{1}{1+1} = \dfrac{1}{2}$.

   Since the third term is $\frac{1}{2}$, the fourth term is $\dfrac{1}{1-\frac{1}{2}} = \dfrac{1}{\frac{1}{2}} = 2$.

   Since the fourth term is 2, the fifth term is $\dfrac{1}{1-2} = \dfrac{1}{-1} = -1$.

   (c) Since the fourth term, 2, is equal to the first term and each term depends only on the previous term, then the sequence of terms repeats every 3 terms.
   That is, the sequence of numbers produced is $2, -1, \frac{1}{2}, 2, -1, \frac{1}{2}, 2, \ldots$.
   Since the terms of the sequence $2, -1, \frac{1}{2}$ repeat every three terms, then we must determine how many groups of three terms there are in the first 2011 terms.
   Since $2011 = 670 \times 3 + 1$, the sequence $2, -1, \frac{1}{2}$ repeats 670 times (giving the first 2010 terms), with the 2011^{th} term being 2.
   That is, there are 671 terms equal to 2 in Fabien's sequence.

   (d) The repeating cycle identified in part (c) has a sum of $2 + (-1) + \frac{1}{2} = \frac{3}{2}$.
   This complete cycle repeats 670 times.
   Thus, the sum of the first 2010 terms in the sequence is $670 \times \frac{3}{2} = 1005$.
   Since the 2011^{th} term is 2, the sum of all terms in Fabien's sequence is $1005 + 2$ or 1007.

2. (a) We organize the possibilities that may appear on the coins in the table below.

'5 coin'	'7 coin'	'10 coin'	Score
0	0	0	0
5	0	0	5
0	7	0	7
0	0	10	10
5	7	0	12
5	0	10	15
0	7	10	17
5	7	10	22

   The other possible scores are $0, 5, 7, 10, 12, 15$, and 22.

   (b) *Solution 1*
   Since the three given scores are different from one another, a different coin must be showing a 0 on each of the three tosses.
   That is, after the three tosses each coin has had its zero side appear once, and its non-zero side appear twice.
   This means that the total of the scores from all three tosses, $60 + 110 + 130 = 300$, represents twice the sum of the number on the non-zero sides of the three coins.
   If twice the sum of the non-zero numbers on the three coins equals 300, then the sum of the non-zero numbers on the three coins is $300 \div 2$ or 150.
   Since the maximum possible score occurs when the non-zero number appears on each of the three coins, then the maximum possible score is 150.

   Solution 2
   Since the three given scores are different from one another, a different coin must be showing a 0 on each of the three tosses.

Let the non-zero number appearing on each of the three coins be a, b and c.
Since exactly one of the three coins shows a zero on each of the three tosses, we may assume without loss of generality that $a + b = 60$, $a + c = 110$, and $b + c = 130$.
Adding the left sides of these three equations gives $a + b + a + c + b + c$ or $2a + 2b + 2c$.
Adding the right sides of the three equations gives $60 + 110 + 130$ or 300.
Since $2a + 2b + 2c = 300$, then $2(a + b + c) = 300$ and so $a + b + c = 150$.
This sum, 150, represents the score when the non-zero number appears on each of the three coins.
Since the maximum possible score occurs when the non-zero number appears on each of the three coins, then the maximum possible score is 150.

(c) We organize the possibilities that may appear on the third coin in the table below, accounting for all of the possible combinations of values from the first two coins:

Appearing on the '25 coin'	Appearing on the '50 coin'	Appearing on the 3^{rd} coin
0	0	$170 - 0 = 170$
25	0	$170 - 25 = 145$
0	50	$170 - 50 = 120$
25	50	$170 - 75 = 95$

The possible non-zero numbers that may appear on the third coin are $170, 145, 120$, and 95.

3. (a) Since $\angle ABP = 90^\circ$, $\triangle ABP$ is a right-angled triangle.
By the Pythagorean Theorem, $BP^2 = AP^2 - AB^2$ or $BP^2 = 20^2 - 16^2$ or $BP^2 = 144$ and so $BP = 12$, since $BP > 0$.
Since $\angle QTP = 90^\circ$, $\triangle QTP$ is a right-angled triangle with $PT = 12$.
Since $PT = BP = 12$, then by the Pythagorean Theorem, $QT^2 = QP^2 - PT^2$ or $QT^2 = 15^2 - 12^2$ or $QT^2 = 81$ and so $QT = 9$, since $QT > 0$.

(b) In triangles PQT and DQS, $\angle PTQ = \angle DSQ = 90^\circ$.
Also, $\angle PQT$ and $\angle DQS$ are vertically opposite angles and are therefore equal.
Since $\angle PTQ = \angle DSQ$, $\angle PQT = \angle DQS$, and the sum of the 3 angles in any triangle is 180°, then the third pair of corresponding angles, $\angle QPT$ and $\angle QDS$, are also equal.
Since the corresponding angles in these two triangles are equal, then $\triangle PQT$ and $\triangle DQS$ are similar triangles.

(c) Since $ABCD$ is a rectangle and TS is perpendicular to BC, then $ABTS$ is also a rectangle.
Thus, $TS = BA = 16$ and $QS = TS - QT = 16 - 9 = 7$.
As shown in part (b), $\triangle PQT$ and $\triangle DQS$ are similar triangles.
Therefore, the ratios of corresponding side lengths in these two triangles are equal.
That is, $\dfrac{SD}{TP} = \dfrac{QS}{QT}$ or $\dfrac{SD}{12} = \dfrac{7}{9}$ or $SD = 12 \times \dfrac{7}{9} = \dfrac{28}{3}$.

(d) *Solution 1*
In $\triangle QAS$ and $\triangle RAD$, $\angle QAS$ and $\angle RAD$ are common (the same) angles and thus are equal.
Since $ABCD$ is a rectangle, $\angle RDA = 90^\circ = \angle QSA$.
Since $\angle QAS = \angle RAD$, $\angle RDA = \angle QSA$, and the sum of the 3 angles in any triangle is 180°, then the third pair of corresponding angles, $\angle SQA$ and $\angle DRA$, are also equal.
Since the corresponding angles in these two triangles are equal, then $\triangle QAS$ and $\triangle RAD$ are similar triangles.

Therefore, the ratios of corresponding side lengths in these two triangles are equal.
That is, $\dfrac{RD}{QS} = \dfrac{DA}{SA}$ or $RD = QS \times \dfrac{DA}{SA}$.
However, $DA = AS + SD = 24 + \dfrac{28}{3} = \dfrac{100}{3}$, and so $RD = 7 \times \dfrac{\left(\frac{100}{3}\right)}{24} = 7 \times \dfrac{100}{72}$ or $RD = \dfrac{175}{18}$.

Since $\triangle QAS$ and $\triangle RAD$ are similar triangles, then $\dfrac{RA}{QA} = \dfrac{RD}{QS}$.
Thus, $RA = QA \times \dfrac{RD}{QS} = 25 \times \dfrac{\left(\frac{175}{18}\right)}{7}$ or $RA = 25 \times \dfrac{25}{18}$, and so $RA = \dfrac{625}{18}$.
Since $QR = RA - QA$, then $QR = \dfrac{625}{18} - 25$ or $QR = \dfrac{625 - 450}{18}$, and so $QR = \dfrac{175}{18}$.
Therefore, $QR = RD$.

Solution 2
In triangles PQA and TQP, the ratios of corresponding side lengths are equal.
That is, $\dfrac{PA}{TP} = \dfrac{PQ}{TQ} = \dfrac{QA}{QP}$ or $\dfrac{20}{12} = \dfrac{15}{9} = \dfrac{25}{15} = \dfrac{5}{3}$.
Therefore, $\triangle PQA$ and $\triangle TQP$ are similar triangles and thus their corresponding angles are equal.
That is, $\angle PQA = \angle TQP = \alpha$.
Since $\angle RQD$ and $\angle PQA$ are vertically opposite angles, then $\angle RQD = \angle PQA = \alpha$.
Since CD and TS are parallel, then by the Parallel Lines Theorem $\angle RDQ = \angle TQP = \alpha$.
Therefore, $\angle RDQ = \angle RQD$ and so $\triangle RQD$ is an isosceles triangle with $QR = RD$.

4. (a) Since $T(4) = 10$ and $T(10) = 55$, then $T(a) = T(10) - T(4) = 45$.
That is, $\dfrac{a(a+1)}{2} = 45$ or $a^2 + a = 90$, and so $a^2 + a - 90 = 0$.
Since $a > 0$ and $(a-9)(a+10) = 0$, then $a = 9$.

(b) The left side of the equation, $T(b+1) - T(b)$, gives $\dfrac{(b+1)(b+2)}{2} - \dfrac{b(b+1)}{2}$, which simplifies to $\dfrac{b^2 + 3b + 2 - b^2 - b}{2}$ or $\dfrac{2b+2}{2}$ or $b+1$.
That is, $b+1$ is equal to $T(x)$, a triangular number.
Since $b > 2011$, we are looking for the the smallest triangular number greater than 2012.
After some trial and error, we observe that $T(62) = 1953$ and $T(63) = 2016$, and so $b + 1 = 2016$ or $b = 2015$ is the smallest value that works.

(c) Since $T(28) = 406$, the second equation gives $c + d + e = 406$ or $e = 406 - (c + d)$.
Next, we simplify the first equation.

$$\begin{aligned} T(c) + T(d) &= T(e) \\ \frac{c(c+1)}{2} + \frac{d(d+1)}{2} &= \frac{e(e+1)}{2} \\ c(c+1) + d(d+1) &= e(e+1) \end{aligned}$$

We now substitute $e = 406 - (c + d)$ into this equation above and simplify.

$$\begin{aligned}
c(c+1) + d(d+1) &= e(e+1) \\
c(c+1) + d(d+1) &= (406 - (c+d))(407 - (c+d)) \\
c^2 + c + d^2 + d &= 406 \times 407 - 406(c+d) - 407(c+d) + (c+d)^2 \\
c^2 + c + d^2 + d &= 406 \times 407 - 813(c+d) + (c+d)^2 \\
c^2 + c + d^2 + d &= 406 \times 407 - 813(c+d) + c^2 + 2cd + d^2 \\
c + d &= 406 \times 407 - 813(c+d) + 2cd \\
2cd &= c + d + 813(c+d) - 406 \times 407 \\
2cd &= 814(c+d) - 406 \times 407 \\
cd &= 407(c+d) - 203 \times 407 \\
cd &= 407(c+d-203),
\end{aligned}$$

as required.

(d) *Solution 1*

Using the result from part (c), we are looking to find all triples (c, d, e) of positive integers, where $c \le d \le e$, such that $cd = 407(c + d - 203)$.
Since the right side of this equation is divisible by 407, then the left side must also be divisible by 407.
Observe that $407 = 37 \times 11$.
Since cd is divisible by 407 and 407 is divisible by 37, then cd is divisible by 37.
Since 37 is a prime number, then one of c or d must be divisible by 37.
Since $c + d + e = 406$ then $d + e \le 406$.
Since $d \le e$, then $d + d \le 406$ or $d \le 203$.
Therefore, $c \le d \le 203$.
Thus, one of c or d is a multiple of 37 that is less than 203.
The largest multiple of 37 less than 203 is $5 \times 37 = 185$.
Next, we try the values $d = 37, 74, 111, 148, 185$ in the equation $cd = 407(c + d - 203)$ to see if we get an integer value for c.
The system of equations that we are solving is symmetric in c and d.
That is, exchanging c and d in the two equations yields the same two equations and thus the same solutions, but with c and d switched.
Therefore, if we happened to get a value of c larger than the value of d that we were trying, then we could just switch them.
In trying the possible values $d = 37, 74, 111, 148, 185$, we only obtain an integer value for c when $d = 185$.
The only triple (c, d, e), where $c \le d \le e$, such that $cd = 407(c+d-203)$ is $(33, 185, 188)$.

Solution 2

Using the result from part (c), we are looking to find all triples (c, d, e) of positive integers, where $c \le d \le e$, such that $cd = 407(c + d - 203)$.
Since the right side of this equation is divisible by 407, then the left side must also be divisible by 407.
Observe that $407 = 37 \times 11$.
Since cd is divisible by 407 and 407 is divisible by 37, then cd is divisible by 37.
Since 37 is a prime number, then one of c or d must be divisible by 37.
Suppose that d is divisible by 37, or that $d = 37n$ for some positive integer n.

(We will consider the possibility that it is c that is divisible by 37 later in the solution.)
Since $c + d + e = 406$ and c, d, e are positive integers, then $1 \le d \le 404$ or $1 \le n \le 10$.
With $d = 37n$ our equation $cd = 407(c + d - 203)$ becomes $37cn = 407(c + 37n - 203)$.
Dividing through by 37, we get $cn = 11(c + 37n - 203)$ or $cn - 11c = 11 \times 37n - 11 \times 203$.
Isolating c in this equation we have $c(n - 11) = 407n - 2233$ or $c = \dfrac{407n - 2233}{n - 11}$.
Since the numerator $407n - 2233$ can be written as $407n - 4477 + 2244$ or $407(n - 11) + 2244$, then we have $c = \dfrac{407(n - 11) + 2244}{n - 11}$ or $c = \dfrac{407(n - 11)}{n - 11} + \dfrac{2244}{n - 11}$ or $c = 407 + \dfrac{2244}{n - 11}$.
Since c is a positive integer, then $n - 11$ must divide 2244.
Since $1 \le n \le 10$, then $-10 \le n - 11 \le -1$.
Thus, the only possibilities for $n - 11$ are $-1, -2, -3, -4$, and -6.
However, of these 5 possibilities only $n - 11 = -6$ gives a positive value for c.
Since $n - 11 = -6$, then $n = 5$, $d = 37 \times 5 = 185$, $c = 33$ and $e = 406 - (c + d) = 188$.
A triple (c, d, e), where $c \le d \le e$, such that $cd = 407(c + d - 203)$ is $(33, 185, 188)$.

Earlier in this solution we made the assumption that d was divisible by 37.
Suppose that it is c that is divisible by 37 or that $c = 37n$ for some positive integer n.
Since $c + d + e = 406$ and c, d, e are positive integers, then $1 \le c \le 404$ or $1 \le n \le 10$.
With $c = 37n$ our equation $cd = 407(c + d - 203)$ becomes $37dn = 407(37n + d - 203)$.
Dividing through by 37, we get $dn = 11(37n + d - 203)$ or $dn - 11d = 11 \times 37n - 11 \times 203$.
Isolating d in this equation we have $d(n - 11) = 407n - 2233$ or $d = \dfrac{407n - 2233}{n - 11}$.
Since the numerator $407n - 2233$ can be written as $407n - 4477 + 2244$ or $407(n - 11) + 2244$, then we have $d = \dfrac{407(n - 11) + 2244}{n - 11}$ or $d = \dfrac{407(n - 11)}{n - 11} + \dfrac{2244}{n - 11}$ or $d = 407 + \dfrac{2244}{n - 11}$.
Since d is a positive integer, then $n - 11$ must divide 2244.
Since $1 \le n \le 10$, then $-10 \le n - 11 \le -1$.
Thus, the only possibilities for $n - 11$ are $-1, -2, -3, -4$, and -6.
However, of these 5 possibilities only $n - 11 = -6$ gives a positive value for d.
Since $n - 11 = -6$, then $n = 5$, $c = 37 \times 5 = 185$, $d = 33$ and $e = 406 - (c + d) = 188$.
Since there is a restriction that $c \le d \le e$, then this solution is not possible.

The only triple (c, d, e), where $c \le d \le e$, such that $cd = 407(c + d - 203)$ is $(33, 185, 188)$.