2011 Galois Contest (Grade 10)

Wednesday, April 13, 2011

1. Jackson gave the following rule to create sequences:
   "If x is a term in your sequence, then the next term in your sequence is $\dfrac{1}{1-x}$."
   For example, Mary starts her sequence with the number 3.
   The second term of her sequence is $\dfrac{1}{1-3} = \dfrac{1}{-2} = -\dfrac{1}{2}$. Her sequence is now $3, -\frac{1}{2}$.
   The third term of her sequence is $\dfrac{1}{1-(-\frac{1}{2})} = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}$. Her sequence is now $3, -\frac{1}{2}, \frac{2}{3}$.
   The fourth term of her sequence is $\dfrac{1}{1-\frac{2}{3}} = \dfrac{1}{\frac{1}{3}} = 3$. Her sequence is now $3, -\frac{1}{2}, \frac{2}{3}, 3$.

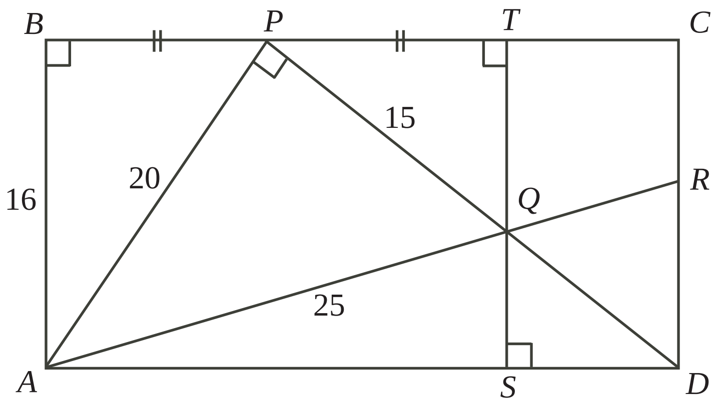
   Fabien starts his sequence with the number 2, and continues using Jackson's rule until the sequence has 2011 terms.

   (a) What is the second term of his sequence?
   (b) What is the fifth term of his sequence?
   (c) How many of the 2011 terms in Fabien's sequence are equal to 2? Explain.
   (d) Determine the sum of all of the terms in his sequence.

2. Alia has a bucket of coins. Each coin has a zero on one side and an integer greater than 0 on the other side. She randomly draws three coins, tosses them and calculates a score by adding the three numbers that appear.

   (a) On Monday, Alia draws coins with a 7, a 5 and a 10. When she tosses them, they show $7, 0$ and 10 for a score of 17. What other scores could she obtain by tossing these same three coins?
   (b) On Tuesday, Alia draws three coins and tosses them three times, obtaining scores of $60, 110$ and 130. On each of these tosses, exactly one of the coins shows a 0. Determine the maximum possible score that can be obtained by tossing these three coins.
   (c) On Wednesday, Alia draws one coin with a 25, one with a 50, and a third coin. She tosses these three coins and obtains a score of 170. Determine all possible numbers, other than zero, that could be on the third coin.

3. In rectangle $ABCD$, P is a point on BC so that $\angle APD = 90^\circ$. TS is perpendicular to BC with $BP = PT$, as shown. PD intersects TS at Q. Point R is on CD such that RA passes through Q. In $\triangle PQA$, $PA = 20$, $AQ = 25$ and $QP = 15$.



(a) Determine the lengths of BP and QT.

(b) Show that $\triangle PQT$ and $\triangle DQS$ are similar. That is, show that the corresponding angles in these two triangles are equal.

(c) Determine the lengths of QS and SD.

(d) Show that $QR = RD$.

4. For a positive integer n, the n^{th} triangular number is $T(n) = \dfrac{n(n+1)}{2}$.
For example, $T(3) = \frac{3(3+1)}{2} = \frac{3(4)}{2} = 6$, so the third triangular number is 6.

(a) There is one positive integer a so that $T(4) + T(a) = T(10)$. Determine a.

(b) Determine the smallest integer $b > 2011$ such that $T(b+1) - T(b) = T(x)$ for some positive integer x.

(c) If $T(c) + T(d) = T(e)$ and $c + d + e = T(28)$, then show that $cd = 407(c + d - 203)$.

(d) Determine all triples (c, d, e) of positive integers such that $T(c) + T(d) = T(e)$ and $c + d + e = T(28)$, where $c \leq d \leq e$.