The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
www.cemc.uwaterloo.ca

Euclid Contest

Tuesday, April 12, 2011

UNIVERSITY OF WATERLOO

WATERLOO MATHEMATICS

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Canadian Institute of Actuaries / Institut canadien des actuaires · Deloitte. · Maplesoft Mathematics • Modeling • Simulation

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 3 marks each. **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this:  .
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this:  .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

The names of some top-scoring students will be published in the Euclid Results on our Web site, http://www.cemc.uwaterloo.ca.

NOTES:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked " " , full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
4. All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, $2\cos(55°)$, etc., rather than as $12.566\ldots$, $4.646\ldots$ or $1.147\ldots$, except where otherwise indicated.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade, and sex, on the Student Information Form, and that you have answered the question about eligibility.

A Note about Writing Solutions

For each problem marked " ", a full solution is required. The solutions that you provide in the answer booklet should be well organized and contain mathematical statements and words of explanation when appropriate. Working out some of the details in rough on a separate piece of paper before writing your finished solution is a good idea. Your final solution should be written so that the marker can understand your approach to the problem and all of the mathematical steps of your solution.

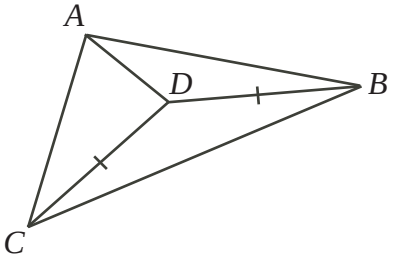
1. (a) If $(x+1)+(x+2)+(x+3)=8+9+10$, what is the value of x?

(b) If $\sqrt{25+\sqrt{x}}=6$, what is the value of x?

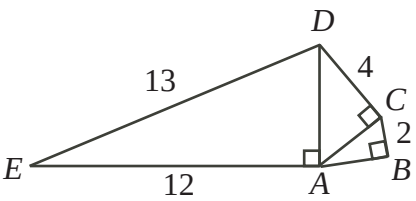
(c) The point $(a,2)$ is the point of intersection of the lines with equations $y=2x-4$ and $y=x+k$. Determine the value of k.

2. (a) An equilateral triangle of side length 1 is cut out of the middle of each side of a square of side length 3, as shown. What is the perimeter of the resulting figure?

(b) In the diagram, $DC=DB$, $\angle DCB=15°$, and $\angle ADB=130°$. What is the measure of $\angle ADC$?



(c) In the diagram, $\angle EAD=90°$, $\angle ACD=90°$, and $\angle ABC=90°$. Also, $ED=13$, $EA=12$, $DC=4$, and $CB=2$. Determine the length of AB.



3. (a) If $2 \leq x \leq 5$ and $10 \leq y \leq 20$, what is the maximum value of $15 - \frac{y}{x}$?

(b) The functions f and g satisfy

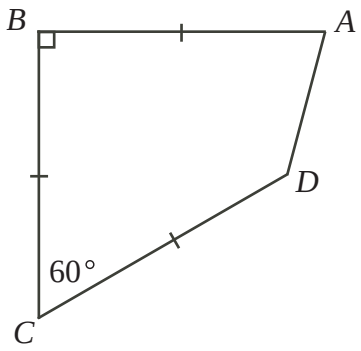
$$\begin{aligned} f(x) + g(x) &= 3x + 5 \\ f(x) - g(x) &= 5x + 7 \end{aligned}$$

for all values of x. Determine the value of $2f(2)g(2)$.

4. (a) Three different numbers are chosen at random from the set $\{1, 2, 3, 4, 5\}$. The numbers are arranged in increasing order. What is the probability that the resulting sequence is an arithmetic sequence?

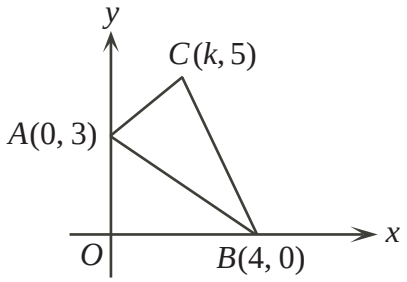
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3,5,7,9 is an arithmetic sequence with four terms.)

(b) In the diagram, $ABCD$ is a quadrilateral with $AB = BC = CD = 6$, $\angle ABC = 90°$, and $\angle BCD = 60°$. Determine the length of AD.



5. (a) What is the largest two-digit number that becomes 75% greater when its digits are reversed?

(b) A triangle has vertices $A(0, 3)$, $B(4, 0)$, $C(k, 5)$, where $0 < k < 4$. If the area of the triangle is 8, determine the value of k.



6. (a) Serge likes to paddle his raft down the Speed River from point A to point B. The speed of the current in the river is always the same. When Serge paddles, he always paddles at the same constant speed. On days when he paddles with the current, it takes him 18 minutes to get from A to B. When he does not paddle, the current carries him from A to B in 30 minutes. If there were no current, how long would it take him to paddle from A to B?

(b) Square $OPQR$ has vertices $O(0, 0)$, $P(0, 8)$, $Q(8, 8)$, and $R(8, 0)$. The parabola with equation $y = a(x - 2)(x - 6)$ intersects the sides of the square $OPQR$ at points K, L, M, and N. Determine all the values of a for which the area of the trapezoid $KLMN$ is 36.

7. 

(a) A 75 year old person has a 50% chance of living at least another 10 years.
A 75 year old person has a 20% chance of living at least another 15 years.
An 80 year old person has a 25% chance of living at least another 10 years.
What is the probability that an 80 year old person will live at least another 5 years?

(b) Determine all values of x for which $2^{\log_{10}(x^2)} = 3(2^{1+\log_{10} x}) + 16$.

8. 

The Sieve of Sundaram uses the following infinite table of positive integers:

$$\begin{array}{ccccc} 4 & 7 & 10 & 13 & \cdots \\ 7 & 12 & 17 & 22 & \cdots \\ 10 & 17 & 24 & 31 & \cdots \\ 13 & 22 & 31 & 40 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

The numbers in each row in the table form an arithmetic sequence. The numbers in each column in the table form an arithmetic sequence. The first four entries in each of the first four rows and columns are shown.

(a) Determine the number in the 50th row and 40th column.

(b) Determine a formula for the number in the Rth row and Cth column.

(c) Prove that if N is an entry in the table, then $2N + 1$ is composite.

9. 

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$.

Suppose that $f(n) = 2n - \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor$ and $g(n) = 2n + \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor$ for each positive integer n.

(a) Determine the value of $g(2011)$.

(b) Determine a value of n for which $f(n) = 100$.

(c) Suppose that $A = \{f(1), f(2), f(3), \ldots\}$ and $B = \{g(1), g(2), g(3), \ldots\}$; that is, A is the range of f and B is the range of g. Prove that every positive integer m is an element of exactly one of A or B.

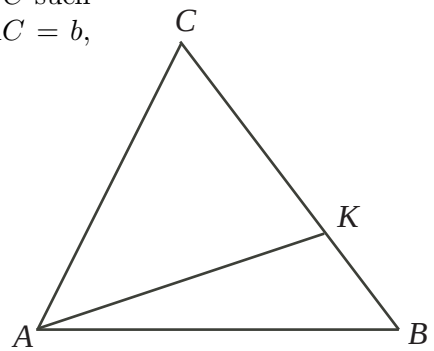
10. 

In the diagram, $2\angle BAC = 3\angle ABC$ and K lies on BC such that $\angle KAC = 2\angle KAB$. Suppose that $BC = a$, $AC = b$, $AB = c$, $AK = d$, and $BK = x$.

(a) Prove that $d = \frac{bc}{a}$ and $x = \frac{a^2 - b^2}{a}$.

(b) Prove that $(a^2 - b^2)(a^2 - b^2 + ac) = b^2c^2$.

(c) Determine a triangle with positive integer side lengths a, b, c and positive area that satisfies the condition in part (b).



The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

For students...

Thank you for writing the 2011 Euclid Contest!
In 2010, more than 16 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2011 Canadian Senior Mathematics Contest, which will be written on November 22, 2011.

Visit our website to find

- Free copies of past contests
- Workshops to help you prepare for future contests
- Information about our publications for mathematics enrichment and contest preparation

For teachers...

Visit our website to

- Obtain information about our 2011/2012 contests, including information about our **new** contests, the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest
- Learn about our face-to-face workshops and our resources
- Find your school contest results

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