The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2011 Canadian Intermediate Mathematics Contest

Tuesday, November 22, 2011
(in North America and South America)

Wednesday, November 23, 2011
(outside of North America and South America)

Solutions

Part A

1. Since the field is rectangular, then its perimeter is $100 + 50 + 100 + 50 = 300$ m.
   Since Joey walks around the perimeter 5 times, then he walks $5 \times 300 = 1500$ m.

   Answer: 1500

2. *Solution 1*
   Since $a + b = 9 - c$ and $a + b = 5 + c$, then $9 - c = 5 + c$.
   Therefore, $9 - 5 = c + c$ or $2c = 4$ or $c = 2$.

   Solution 2
   Since $a + b = 9 - c$, then $c = 9 - a - b$.
   Since $a + b = 5 + c$, then $a + b = 5 + 9 - a - b$ or $2a + 2b = 14$ and so $a + b = 7$.
   Since $c = 9 - a - b = 9 - (a + b)$, then $c = 9 - 7 = 2$.

   Answer: 2

3. *Solution 1*
   Suppose that Ophelia works for a total of n weeks.
   Then she earns \$51 for 1 week and \$100 a week for $(n - 1)$ weeks.
   Since her average weekly pay is \$93, then $\dfrac{51 + (n-1) \times 100}{n} = 93$.
   Therefore,

$$\begin{aligned} 51 + 100(n-1) &= 93n \\ 51 + 100n - 100 &= 93n \\ 100n - 49 &= 93n \\ 100n - 93n &= 49 \\ 7n &= 49 \\ n &= 7 \end{aligned}$$

   Thus, Ophelia works for a total of 7 weeks.

   Solution 2
   We make a table that lists what Ophelia earns each week, the total that she has earned at the end of that week, and her average weekly pay so far:

Week	Pay this week (\$)	Total pay so far (\$)	Average pay so far (\$)
1	51	51	51.00
2	100	151	$151 \div 2 = 75.50$
3	100	251	$251 \div 3 \approx 83.67$
4	100	351	$351 \div 4 = 87.75$
5	100	451	$451 \div 5 = 90.20$
6	100	551	$551 \div 6 \approx 91.83$
7	100	651	$651 \div 7 = 93.00$

   Therefore, Ophelia's average weekly pay is \$93 after she has worked for 7 weeks.

   Answer: 7

4. When the red die is rolled, there are 6 equally likely outcomes. Similarly, when the blue die is rolled, there are 6 equally likely outcomes.
Therefore, when the two dice are rolled, there are $6 \times 6 = 36$ equally likely outcomes for the combination of the numbers on the top face of each. (These outcomes are Red 1 and Blue 1, Red 1 and Blue 2, Red 1 and Blue 3, ..., Red 6 and Blue 6.)
The chart below shows these possibilities along with the sum of the numbers in each case:

		Blue Die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Red	3	4	5	6	7	8	9
Die	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Since the only perfect squares between 2 and 12 are 4 (which equals 2^2) and 9 (which equals 3^2), then 7 of the 36 possible outcomes are perfect squares.
Since each entry in the table is equally likely, then the probability that the sum is a perfect square is $\frac{7}{36}$.

Answer: $\frac{7}{36}$

5. Since $\triangle ADP$ is isosceles and $\angle ADP = 2x^\circ$, then $\angle DAP = \angle DPA = \frac{1}{2}(180^\circ - 2x^\circ) = 90^\circ - x^\circ$.
Similarly, since $\triangle CDP$ is isosceles and $\angle CDP = 2x^\circ$, then $\angle DCP = \angle DPC = 90^\circ - x^\circ$.
Since $\triangle BAP$ is isosceles and $\angle BAP = (x+5)^\circ$, then $\angle BPA = \frac{1}{2}(180^\circ - (x+5)^\circ) = \frac{1}{2}(175-x)^\circ$.
Since the measures of the four angles around point P must add to 360°, then

$$\begin{aligned}
\angle DPC + \angle DPA + \angle BPA + \angle BPC &= 360^\circ \\
(90^\circ - x^\circ) + (90^\circ - x^\circ) + \tfrac{1}{2}(175 - x)^\circ + (10x - 5)^\circ &= 360^\circ \\
(90 - x) + (90 - x) + \tfrac{1}{2}(175 - x) + (10x - 5) &= 360 \\
(180 - 2x) + (180 - 2x) + (175 - x) + (20x - 10) &= 720 \\
15x + 525 &= 720 \\
15x &= 195 \\
x &= 13
\end{aligned}$$

Therefore, $x = 13$.

Answer: 13

6. *Solution 1*
We rewrite 6^{16} as $(2 \cdot 3)^{16}$, which equals $2^{16}3^{16}$.
If d is a positive integer divisor of 6^{16}, it cannot have more than 16 factors of 2 or of 3, and cannot include any other prime factors. Therefore, if d is a positive integer divisor of 6^{16}, then $d = 2^m3^n$ for some integers m and n with $0 \leq m \leq 16$ and $0 \leq n \leq 16$.
Since there are 17 possible values for each of m and n, then there are $17 \times 17 = 289$ possible positive integer divisors d.
Since 6^{16} is a perfect square ($6^8 \times 6^8$), then every positive integer divisor other than its square

root (6^8) can be paired with a different positive integer such that the product of the pair is 6^{16}. (For example, $2^4 3^{13}$ can be paired with $2^{12}3^3$ since $2^4 3^{13} \times 2^{12}3^3 = 2^{4+12}3^{13+3} = 2^{16}3^{16}$.)
Since there are 289 positive integer divisors in total, then there are $\frac{1}{2}(289-1) = 144$ of these divisor pairs whose product is 6^{16}.
To multiply all of the positive integer divisors together, we can multiply the product of the 144 pairs with the remaining divisor (6^8).
Therefore, the product of all of the positive integer divisors is $(6^{16})^{144} \cdot 6^8 = 6^{16(144)+8} = 6^{2312}$.
Thus, $k = 2312$.

Solution 2
We rewrite 6^{16} as $(2 \cdot 3)^{16}$, which equals $2^{16}3^{16}$.
If d is a positive integer divisor of 6^{16}, it cannot have more than 16 factors of 2 or of 3, and cannot include any other prime factors. Therefore, if d is a positive integer divisor of 6^{16}, then $d = 2^m 3^n$ for some integers m and n with $0 \leq m \leq 16$ and $0 \leq n \leq 16$.
Consider the 17 divisors of the form $2^0 3^n$ for $n = 0$ to $n = 16$.
These divisors are $3^0, 3^1, 3^2, \ldots, 3^{15}, 3^{16}$. Their product is

$$3^0 \cdot 3^1 \cdot 3^2 \cdot \cdots \cdot 3^{15} \cdot 3^{16} = 3^{0+1+2+\cdots+15+16}$$

By the Useful Fact given, $0+1+2+\cdots+15+16 = 1+2+\cdots+15+16 = \frac{1}{2}(16)(17) = 136$, and so the product of these divisors is 3^{136}.
Next, consider the 17 divisors of the form $2^1 3^n$ for $n = 0$ to $n = 16$.
These divisors are $2^1 3^0, 2^1 3^1, 2^1 3^2, \ldots, 2^1 3^{15}, 2^1 3^{16}$. Their product is

$$2^1 3^0 \cdot 2^1 3^1 \cdot 2^1 3^2 \cdot \cdots \cdot 2^1 3^{15} \cdot 2^1 3^{16} = 2^{17} 3^{0+1+2+\cdots+15+16} = 2^{17} 3^{136}$$

In general, consider the 17 divisors of the form $2^m 3^n$, for a fixed value of m and for $n = 0$ to $n = 16$. These divisors are $2^m 3^0, 2^m 3^1, 2^m 3^2, \ldots, 2^m 3^{15}, 2^m 3^{16}$. Their product is

$$2^m 3^0 \cdot 2^m 3^1 \cdot 2^m 3^2 \cdot \cdots \cdot 2^m 3^{15} \cdot 2^m 3^{16} = 2^{17m} 3^{0+1+2+\cdots+15+16} = 2^{17m} 3^{136}$$

To obtain the overall product of all of the positive integer divisors, we can multiply together the products of these sets of 17 divisors.
This product is

$$\begin{aligned} & 2^0 3^{136} \cdot 2^{17(1)} 3^{136} \cdot 2^{17(2)} 3^{136} \cdot \cdots \cdot 2^{17(15)} 3^{136} \cdot 2^{17(16)} 3^{136} \\ &= 2^{17(0+1+2+\cdots+15+16)} (3^{136})^{17} \\ &= 2^{17(136)} 3^{17(136)} \\ &= 2^{2312} 3^{2312} \\ &= 6^{2312} \end{aligned}$$

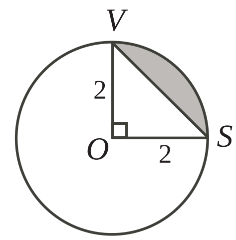
Therefore, the desired product is 6^{2312}, and so $k = 2312$.

Answer: 2312

Part B

1. (a) Since the circle has radius 2, its area is $\pi 2^2 = 4\pi$.
The central angle for the shaded sector is 90°, which is one-quarter of the full circle (360°).
Therefore, the shaded area is one-quarter of the area of the full circle, or $\frac{1}{4} \times 4\pi = \pi$.

(b) We label the diagram as shown.



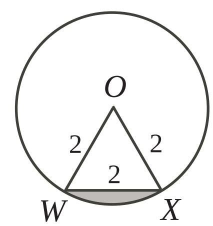
The area of the shaded region equals the area of the sector OSV minus the area of $\triangle OSV$.
From (a), the area of the sector is π.
Since the triangle is right-angled and has base and height both equal to 2, its area is $\frac{1}{2}(2)(2) = 2$.
Therefore, the area of the shaded region is $\pi - 2$.

(c) Since $\triangle PQR$ is equilateral, then altitude PT cuts base QR into two equal pieces.
Since $QR = 2$, then $QT = TR = 1$.
By the Pythagorean Theorem in $\triangle PTQ$, we have

$$PT = \sqrt{PQ^2 - QT^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$$

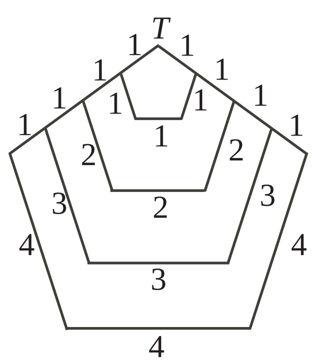
since $PT > 0$.
Therefore, $\triangle PQR$ has base $QR = 2$ and height $PT = \sqrt{3}$.
Thus, the area of $\triangle PQR$ is $\frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$.

(d) We label the diagram as shown.



The area of the shaded region equals the area of the sector WOX minus the area of $\triangle OWX$.
From (c), the area of $\triangle OWX$ is $\sqrt{3}$, since $\triangle OWX$ is congruent to $\triangle PQR$.
Since $\triangle OWX$ is equilateral, then $\angle WOX = 60^\circ$; since $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$, then this is one-sixth of the full circle.
Therefore, the area of sector WOX is one-sixth of the area of the entire circle, or $\frac{1}{6}(4\pi) = \frac{2}{3}\pi$.
Thus, the area of the shaded region is $\frac{2}{3}\pi - \sqrt{3}$.

2. (a) Figure 4 is shown below. Its ink length can be found in a number of ways, including by labeling the various segment lengths as shown:



The total ink length is thus $11 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 = 38$.
(We could instead use a similar method to the methods from (b) or (c) to calculate this.)

(b) We calculate the difference between these ink lengths by mentally starting with Figure 8 and determining what needs to be added to Figure 8 to produce Figure 9.
Figure 9 is a copy of Figure 8 with a regular pentagon of side length 9 added and overlapping segments removed.
A regular pentagon with side length 9 has an ink length of $5 \cdot 9 = 45$.
The overlapping segments that must be removed are the two edges coming out of T in Figure 8, which each have length 8.
Therefore, the difference in ink length is $5 \cdot 9 - 2 \cdot 8 = 45 - 16 = 29$.

(c) *Solution 1*
Figure 100 consists of 100 regular pentagons of side lengths 1 to 100, inclusive, each sharing a common vertex at T and with edges on either side of T overlapping.
Figure 100 can be constructed using the bottom three edges of each of these pentagons and the top two edges of the pentagon of side length 100.
Its ink length is thus

$$\begin{aligned} &3(1) + 3(2) + \cdots + 3(99) + 3(100) + 2(100) \\ &\qquad = 3(1 + 2 + \cdots + 99 + 100) + 2(100) \\ &\qquad = 3(\tfrac{1}{2}(100)(101)) + 200 \\ &\qquad = 150(101) + 200 \\ &\qquad = 15150 + 200 \\ &\qquad = 15350 \end{aligned}$$

Therefore, the ink length of Figure 100 is 15 350.

Solution 2
The ink length of Figure 100 can be calculated by starting with Figure 99 and determining how much ink length needs to be added.
Similarly, the ink length of Figure 99 can be calculated by starting with Figure 98 and determining how much ink length needs to be added.
In general, the ink length of Figure k can be calculated by starting with Figure $(k-1)$ and determining how much ink length needs to be added.
Using this process, we can start with Figure 1, add to this to get Figure 2, add to this to get Figure 3, and so on, all of the way up to Figure 100.
To obtain the difference in ink length between Figure k and Figure $(k-1)$, we can model what we did in (b):

> We calculate the difference between these ink lengths by mentally starting with Figure $(k-1)$ and determining what needs to be added to Figure $(k-1)$ to produce Figure k.
> Figure k is a copy of Figure $(k-1)$ with a regular pentagon of side length k added and overlapping segments removed.
> A regular pentagon with side length k has an ink length of $5k$.
> The overlap segments that must be removed are the two edges coming out of T in Figure $(k-1)$, which each have length $(k-1)$.
> Therefore, the difference in ink length is $5k - 2(k-1) = 3k + 2$.

Therefore, to get from the ink length of Figure 1 (which is 5) to the ink length of Figure 100 we need to add each of the differences $3k+2$ (calculated above) from $k = 2$ to $k = 100$.

Therefore, the ink length of Figure 100 is

$$\begin{aligned} &5 + (3(2)+2) + (3(3)+2) + \cdots + (3(99)+2) + (3(100)+2) \\ &\quad = (3(1)+2) + (3(2)+2) + (3(3)+2) + \cdots + (3(99)+2) + (3(100)+2) \\ &\quad = 3(1+2+3+\cdots+99+100) + 100(2) \\ &\quad = 3(\tfrac{1}{2}(100)(101)) + 200 \\ &\quad = 150(101) + 200 \\ &\quad = 15150 + 200 \\ &\quad = 15350 \end{aligned}$$

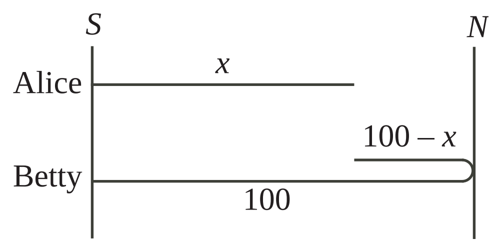
Therefore, the ink length of Figure 100 is 15 350.

3. (a) *Solution 1*

Since Betty swims 600 m in the time that Alice swims 400 m, then the ratio of their speeds is $600 : 400 = 3 : 2$.
Therefore, at any instant in time, the ratio of the distance that Betty has swum to the distance that Alice has swum will equal $3 : 2$.
Since Alice swims 4 lengths of the pool, let us check for a crossing during each of these lengths.

Alice's First Length

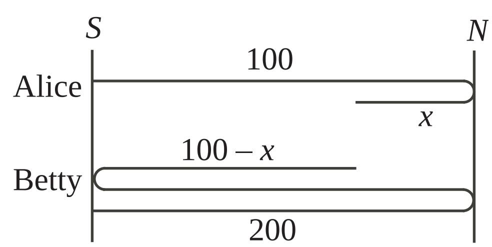
Suppose that Alice and Betty cross during this length.
Here, Alice is swimming towards N and has swum x m, where x is between 0 and 100.
Since Betty swims $\frac{3}{2}$ as far as Alice in the same time, then Betty has swum between $\frac{3}{2}\times 0 = 0$ m and $\frac{3}{2}\times 100 = 150$ m.
For Alice and Betty to cross, Betty must be swimming towards S, and so is on her second length. She has swum 100 m (a full length) plus $(100 - x)$ m to meet Alice.



Since the ratio of their distances is $3 : 2$, then $\dfrac{200-x}{x} = \dfrac{3}{2}$ and so $400 - 2x = 3x$ or $5x = 400$, and so $x = 80$.
This is in the correct range, so Alice and Betty cross during Alice's first length.

Alice's Second Length

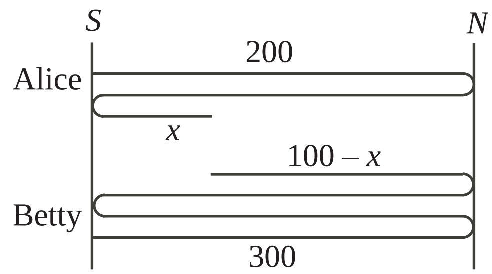
Suppose that Alice and Betty cross during this length.
Here, Alice is swimming towards S and has swum between 100 and 200 m. We represent her distance as $(100 + x)$ m, where x is between 0 and 100. From the ratio of their speeds, Betty has swum between 150 m and 300 m.
For Alice and Betty to cross, Betty must be swimming towards N, and so is on her third length. She has swum 200 m (two full lengths) plus $(100-x)$ m to meet Alice.
(This is a total of $200 + (100 - x) = 300 - x$ m.)



Since the ratio of their distances is $3 : 2$, then $\dfrac{300-x}{100+x} = \dfrac{3}{2}$ and so $600 - 2x = 300 + 3x$

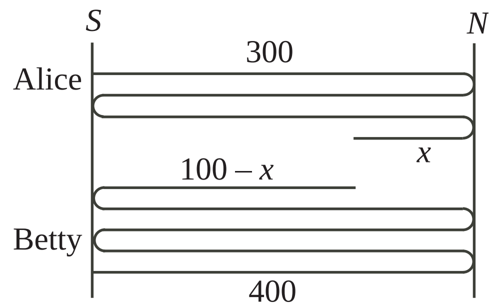
or $5x = 300$, and so $x = 60$.
This is in the correct range (Alice has swum 160 m and Betty has swum 240 m), so Alice and Betty cross during Alice's second length.

Alice's Third Length
Suppose that Alice and Betty cross during this length. Here, Alice is swimming towards N and has swum between 200 and 300 m. We represent her distance as $(200 + x)$ m, where x is between 0 and 100. From the ratio of their speeds, Betty has swum between 300 m and 450 m. For Alice and Betty to cross, Betty must be swimming towards S, and so is on her fourth length. We represent the distance that she has swum at this potential crossing as $(400 - x)$ m.



Thus, $\dfrac{400 - x}{200 + x} = \dfrac{3}{2}$ and so $800 - 2x = 600 + 3x$ or $5x = 200$, and so $x = 40$.
This is in the correct range (Alice has swum 240 m and Betty has swum 360 m), so Alice and Betty cross during Alice's third length.

Alice's Fourth Length
Suppose that Alice and Betty cross during this length. Here, Alice is swimming towards S and has swum between 300 and 400 m. We represent her distance as $(300 + x)$ m, where x is between 0 and 100. From the ratio of their speeds, Betty has swum between 450 m and 600 m. For Alice and Betty to cross, Betty must be swimming towards N, and so is on her fifth length. We represent the distance that she has swum at this potential crossing as $(500 - x)$ m.



Thus, $\dfrac{500 - x}{300 + x} = \dfrac{3}{2}$ and so $1000 - 2x = 900 + 3x$ or $5x = 100$, and so $x = 20$.
This is in the correct range (Alice has swum 320 m and Betty has swum 480 m), so Alice and Betty cross during Alice's fourth length.

Therefore, Alice and Betty cross 4 times (once during each of Alice's lengths of the pool) before finally crossing at S.

Solution 2
Consider the given situation.
Since Betty swims 600 m in the time that Alice swims 400 m, then the ratio of their speeds is $600 : 400 = 3 : 2$.
Therefore, in the time that Alice swims d m, Betty will swim $\frac{3}{2}d$ m.
At this final point, they have swum a total of $400 + 600 = 1000$ m.
By Important Fact #1 below, the possible previous total distances at which they could cross are 200 m, 400 m, 600 m, and 800 m.
We do not yet know that each of these will yield a crossing, so we check each:

- If Alice has swum d m (and so Betty has swum $\frac{3}{2}d$ m) and they have swum a total of 200 m, then $d + \frac{3}{2}d = 200$ or $\frac{5}{2}d = 200$ or $d = 80$. In this case, Alice has swum 80 m and Betty has swum $\frac{3}{2} \times 80 = 120$ m. Since Alice is swimming towards N and Betty towards S at this point and each of them is 20 m from N, then they are at the same point in the pool and swimming in opposite directions, so this is a crossing.

- In the case of $d + \frac{3}{2}d = 400$, then $\frac{5}{2}d = 400$ or $d = 160$. In this case, Alice has swum 160 m and Betty has swum $\frac{3}{2} \times 160 = 240$ m. Since Alice is swimming towards S and Betty towards N at this point and they are each 40 m from S, then this is a crossing.
- In the case of $d + \frac{3}{2}d = 600$, then $\frac{5}{2}d = 600$ or $d = 240$. In this case, Alice has swum 240 m and Betty has swum $\frac{3}{2} \times 240 = 360$ m. Since Alice is swimming towards N and Betty towards S at this point and they are each 40 m from S, then this is a crossing.
- In the case of $d + \frac{3}{2}d = 800$, then $\frac{5}{2}d = 800$ or $d = 320$. In this case, Alice has swum 320 m and Betty has swum $\frac{3}{2} \times 320 = 480$ m. Since Alice is swimming towards S and Betty towards N at this point and they are each 20 m from N, then this is a crossing.

Therefore, before the final point, they cross 4 times.

<u>Important Fact #1</u>
Call the two swimmers K and L.
If K and L cross at S, then they have both swum an even number of lengths of the pool, so the distance that each has swum is an even multiple of 100 m. Therefore, if they cross at S, the sum of the distances that they have swum is an even multiple of 100 m.
If K and L cross at N, then they have both swum an odd number of lengths of the pool, so the distance that each has swum is an odd multiple of 100 m. Therefore, if they cross at N, the sum of the distances that they have swum is again an even multiple of 100 m (because an odd multiple of 100 plus an odd multiple of 100 is an even multiple of 100).
If K and L cross at a point P between S and N, what happens? If P is d m from S, then

- one of them is swimming from S and so has swum an even number of lengths (thus an even multiple of 100 m) plus an additional d m from P to S, and
- the other is swimming from N and so has swum an odd number of lengths (thus an odd multiple of 100 m) plus an additional $(100 - d)$ m from N to P.

Therefore, the sum of the distances that they have swum is an even multiple of 100 m plus an odd multiple of 100 m plus d m plus $(100 - d)$ m, which is an even multiple of 100 m plus an odd multiple of 100 m plus 100 m. In total, this is an even multiple of 100 m.
In summary, if they cross at a point, then their total distance swum so far is an even multiple of 100 m (which is a multiple of 200 m).

(b) If Charlie and David first cross when Charlie has swum 90 m, then Charlie is swimming towards N at this point, so David is swimming towards S and has swum 10 m from N. Since this is the first time that they cross, David must have swum only one complete length plus the additional 10 m, for 110 m in total. (If David had swum a larger number of complete lengths, he would have to have crossed Charlie earlier while swimming from N to S.)
Since David swims 110 m in the time that Charlie swims 90 m, then the ratio of their speeds is $110 : 90 = 11 : 9$.
Therefore, in the time that Charlie swims $9d$ m, David swims $11d$ m.
Charlie and David arriving together at S happens exactly when each has swum a multiple of 200 m.
Now Charlie and David arrive together at S when Charlie has swum 1800 m and David has swum 2200 m, since $2200 : 1800 = 11 : 9$. This is the first time that they arrive together at S, since no positive multiple of 200 smaller than 1800 when multiplied by $\frac{11}{9}$

gives another multiple of 200. (This is because no positive multiple of 200 smaller than 1800 is a multiple of 9.)
Their total distance swum at this point is $1800 + 2200 = 4000$ m.
By Important Fact #1 above, the possible earlier points at which they cross are the points at which their total distance swum is a multiple of 200 m. There are 19 such points corresponding to total distances 200 m, 400 m, ..., 3600 m, and 3800 m. (Note that $3800 = 19 \times 200$.)
By Important Fact #2 below, Charlie and David do in fact cross at each of these points.
Therefore, they cross 18 more times (after the first crossing) before arriving together at S for the first time.

Important Fact #2
Call the two swimmers K and L.
Suppose that they are at an instance when their total distance swum is a multiple of 200 m. We want to explain why they must cross when this is true.
Suppose that K is at one end of the pool at this instance. If K is at N, then K has swum an odd multiple of 100 m. Since K and L together have swum a multiple of 200 m (that is, an even multiple of 100 m), then L must also have swum an odd multiple of 100 m, and so L is also at N. Thus, K and L cross at this point.
Similarly, if K is at S, then L will also be at S and thus they cross, by definition.
Suppose then that K is not at an end of the pool when this happens. Then the distance that K has swum is not a multiple of 100 m, and so we can write the distance as $(100x + y)$ m with x a non-negative integer and y satisfying $0 < y < 100$. (In other words, K has swum x lengths and y m more.)
Similarly, write the distance that L has swum as $(100w + z)$ m with w a non-negative integer and z satisfying $0 \leq z < 100$.
Since the sum of their distances swum is a positive multiple of 200 m, then we can write $(100x + y) + (100w + z) = 200p$ for some positive integer p.
We rewrite this as $y + z = 200p - 100x - 100w = 100(2p - x - w)$. This shows us that the right side is a multiple of 100, so the left side must be too. Since y and z are each between 0 and 100, then $y + z$ is strictly in between 0 and 200, so $y + z = 100$.
From this, we get $100 = 100(2p - x - w)$ or $1 = 2p - x - w$ or $x + w + 1 = 2p$.
Since the right side is even, then the left side is even, and so $x + w$ is odd.
This means that one of x and w is odd and the other is even. This means that K and L are swimming in opposite directions at this instant (one is coming from N and one is coming from S).
Since $y + z = 100$, then the partial pool length that each has completed at this instance add up to a full pool length; since they are swimming in opposite directions, then K and L are at the same place in the pool.
Since K and L are at the same place and swimming in opposite directions, then they cross at this instant.
Therefore, if the total distance swum is a multiple of 200 m, then the swimmers cross.

(c) *Solution 1*
By Important Fact #1, if Evan and Farouq cross at a point, then the sum of the distances that they have swum is a multiple of 200 m.
By Important Fact #2, at every point where the sum of the distances that they have swum is a multiple of 200 m, Evan and Farouq cross.

Thus, between two consecutive instances when Evan and Farouq cross, their total distance travelled increases by 200 m.
Because their speeds are constant, the amount of time that it takes for the total distance to increase by 200 m does not depend on which particular consecutive crossings are involved.
Therefore, the time between two consecutive instances when they cross is constant.

Solution 2
Suppose that Farouq's speed is v_F m/s and Evan's speed is v_E m/s.
Consider two consecutive instances when they cross.
At the first instance, suppose they have swum a combined distance of $200n$ m, for some positive integer n. (We know that the total distance swum is a multiple of 200 m by Important Fact #1.)
Then at the next instance when they cross, their total distance swum is $(200n + 200)$ m. (We know that the total distance swum is the next multiple of 200 m by Important Fact #2.)
Suppose that they have swum for a length of time t s at the first instance and T s at the second instance.
Since speed multiplied by time gives distance, then at the first instance, $v_Ft + v_Et = 200n$ and so $t(v_F + v_E) = 200n$.
At the next instance, $v_FT + v_ET = 200n + 200$ and so $T(v_F + v_E) = 200n + 200$.
Subtracting the first from the second, we get $T(v_F + v_E) - t(v_F + v_E) = 200$.
Factoring, we get $(T - t)(v_F + v_E) = 200$ and so $T - t = \dfrac{200}{v_F + v_E}$.
This is the length of time between an arbitrary pair of consecutive instances when they cross.
Since v_F and v_E are constant and $T - t$ depends only on v_F and v_E, then it is constant, as required.