Canadian Mathematics Competition

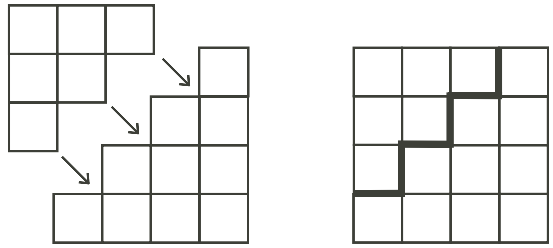
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Fryer Contest

Friday, April 9, 2010

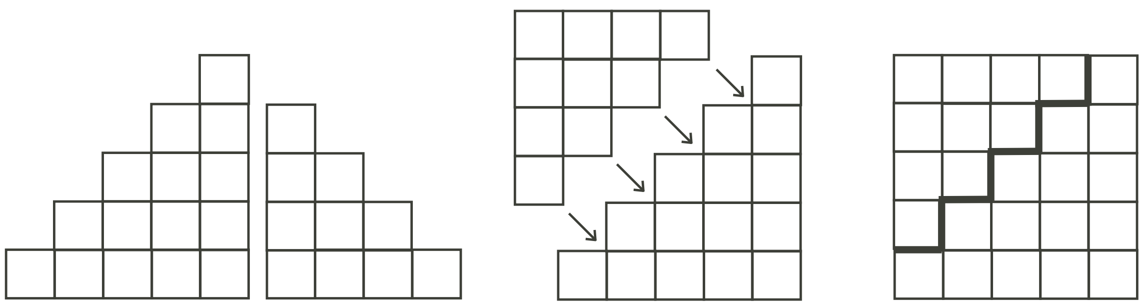
Solutions

1. (a) The piece on the right can be repositioned to rest on the left piece as shown.
The resulting arrangement is a 4 by 4 square having 16 tiles.



(b) *Solution 1*
The top 4 rows of Figure 5 are identical to the 4 rows that form Figure 4. The bottom row of Figure 5 has two tiles more than the bottom row of Figure 4, or $7 + 2 = 9$ tiles. Thus, Figure 5 has $1 + 3 + 5 + 7 + 9 = 25$ tiles.

Solution 2
The bottom (5th) row of Figure 5 has two tiles more than the bottom row of Figure 4, or $7 + 2 = 9$ tiles. Using the method of part (a), Figure 5 can be cut into two pieces and repositioned as shown. The resulting arrangement is a 5 by 5 square having 25 tiles.



(c) We count the number of tiles in the bottom row of each of the first 5 figures and list the results in the following table:

Figure Number	Number of tiles in the bottom row
1	1
2	3
3	5
4	7
5	9

Since the number of tiles in the bottom row of any figure is two more than the number of tiles in the bottom row of the previous figure, we can continue the table.

Figure Number	Number of tiles in the bottom row
6	11
7	13
8	15
9	17
10	19

Therefore, Figure 10 has 19 tiles in its bottom row.

(d) *Solution 1*

The top 9 rows of Figure 11 are identical to the 9 rows that form Figure 9.
Since Figure 11 has 11 rows, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is equal to the sum of the number of tiles in the 10th and 11th rows of Figure 11.
The number of tiles in the 10th row of Figure 11 is equal to the number of tiles in the 10th (bottom) row of Figure 10, or 19.
From part (c), we also know that the bottom row of Figure 11 has 2 tiles more than the bottom row of Figure 10, or $19 + 2 = 21$ tiles.
Therefore, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is $19 + 21 = 40$ tiles.

Solution 2

Using the method of part (a), it can be shown that Figure 11 can be cut into 2 pieces and rearranged to form an 11 by 11 square having $11 \times 11 = 121$ tiles.
Similarly, Figure 9 can be shown to have $9 \times 9 = 81$ tiles.
Therefore, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is $121 - 81 = 40$ tiles.

2. (a) The average of any set of integers is equal to the sum of the integers divided by the number of integers.
Thus, the average of the integers is $\frac{71+72+73+74+75}{5} = \frac{365}{5} = 73$.

(b) (i) Simplifying, $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$.

(ii) Since the sum of the 5 consecutive integers is $5n + 10$, the average of these integers is $\dfrac{5n + 10}{5} = n + 2$.
If n is an even integer, then $n + 2$ is an even integer.
If n is an odd integer, then $n + 2$ is an odd integer.
Thus for the average $n + 2$ to be odd, the integer n must be odd.

(c) Simplifying, $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) = 6n + 15$.
Since the sum of the 6 consecutive integers is $6n + 15$, the average of these integers is $\dfrac{6n + 15}{6} = n + \frac{15}{6} = n + \frac{5}{2}$.
For every integer n, $n + \frac{5}{2}$ is never an integer.
Therefore, the average of six consecutive integers is never an integer.

3. (a) Travelling at a speed of 60 km/h for 9 hours, Train 1 travels a distance of $60 \times 9 = 540$ km.
If the distance from Amville to Batton is d, then $\frac{2}{3}d = 540$ and $d = \frac{540(3)}{2} = 810$ km.
Therefore, the distance from Amville to Batton is 810 km.

(b) The distance from Batton to Amville is 810 km.
Two-thirds of the distance from Batton to Amville is $\frac{2}{3} \times 810 = 540$ km.
Train 2 travelled the 540 km in 6 hours.
Thus, Train 2 travels at a constant speed of $\frac{540}{6} = 90$ km/h.

(c) Let t hours be the time that it takes Train 1 to travel from Amville to Cuford.
Since Train 1 travels at 60 km/h, the distance from Amville to Cuford is $60t$ km.
Train 2 leaves Batton $3\frac{1}{2}$ hours after Train 1 leaves Amville.
Since the trains arrive at Cuford at the same time, Train 2 takes $\left(t - 3\frac{1}{2}\right)$ hours to travel from Batton to Cuford.

Since Train 2 travels at 90 km/h, the distance from Batton to Cuford is $90(t - 3\frac{1}{2})$ km.
The distance from Amville to Cuford added to the distance from Cuford to Batton is equal to the distance from Amville to Batton, 810 km.
Thus, $60t + 90(t - 3\frac{1}{2}) = 810$ or $60t + 90t - 315 = 810$ or $150t = 1125$ and $t = 7\frac{1}{2}$ hours.
Therefore, Train 1 needed $7\frac{1}{2}$ hours to travel from Amville to Cuford.
Since Train 1 arrived at Cuford at 9:00 p.m., Train 1 left Amville at 1:30 p.m.

4. (a) A palindrome less than 1000 must be either 1, 2 or 3 digits in length.
We consider each of these 3 possibilities as separate cases below.

<u>Case 1: one-digit palindromes</u>
All of the positive integers from 1 to 9 are palindromes since they are the same when read forwards or backwards. Thus, there are 9 one-digit palindromes.
<u>Case 2: two-digit palindromes</u>
We are looking for all of the palindromes from the positive integers 10 to 99.
To be a two-digit palindrome, the first digit must equal the second digit.
Thus, there are only 9 two-digit palindromes; $\{11, 22, 33, 44, 55, 66, 77, 88, 99\}$.
<u>Case 3: three-digit palindromes</u>
All three-digit palindromes are of the form aba, for integers a and b, where $1 \leq a \leq 9$ and $0 \leq b \leq 9$.
That is, to create a three-digit palindrome the hundreds digit and the units digit must be equal to each other, but not equal to zero.
Since the hundreds digit a can be any of the positive integers from 1 to 9, there are 9 choices for the hundreds digit a.
Since the tens digit b can be any of the integers from 0 to 9, there are 10 choices for b.
The final digit (the units digit) must equal the hundreds digit, thus there is no choice for the units digit since it has already been chosen.
In total, there are 9×10 choices for a and b and therefore, 90 three-digit palindromes.

There are 9 one-digit palindromes, 9 two-digit palindromes and 90 three-digit palindromes.
In total, there are $9 + 9 + 90 = 108$ palindromes less than 1000.

(b) All seven-digit palindromes are of the form $abcdcba$, for integers a, b, c, and d, where $1 \leq a \leq 9$ and $0 \leq b, c, d \leq 9$.
Since the last 3 digits cba are determined by the first three digits abc, the number of seven-digit palindromes is dependent on the number of choices for the first 4 digits $abcd$ only. There are 9 choices for the first digit a, 10 choices for the second digit b, 10 choices for the third digit c, and 10 choices for the fourth digit d.
In total, there are $9 \times 10 \times 10 \times 10$ choices for $abcd$ and therefore, 9000 seven-digit palindromes.

(c) The seven-digit palindromes between 1000000 and 2000000 begin with a 1, and are of the form $1bcdcb1$.
As in part (b), there are 10 choices for each of b, c and d, and therefore $10 \times 10 \times 10 = 1000$ palindromes between 1000000 and 2000000.
In the increasing list of seven-digit palindromes, these are the first 1000 palindromes.
Similarly, the next 1000 palindromes lie between 2000000 and 3000000, and the next 1000 lie between 3000000 and 4000000.
Thus, the 2125th palindrome must lie between 3000000 and 4000000 and must be of the form $3bcdcb3$.
The smallest palindromes of the form $3bcdcb3$ are those of the form $30cdc03$.
Since there are 10 possibilities for each of c and d, then there are $10 \times 10 = 100$ such

palindromes.
The largest of these (3099903) is the 2100th palindrome in the list.
Similarly, there are 100 palindromes of the form $31cdc13$.
The 2125th palindrome will thus be of this form.
The smallest palindromes of the form $31cdc13$ are those of the form $310d013$.
Since there are 10 possibilities for d, then there are 10 such palindromes.
Similarly, there are 10 palindromes of the form $311d113$.
The largest of these (3119113) is the 2120th in the list.
Also, there are 10 palindromes of the form $312d213$.
The 2125th palindrome in the entire list is the 5th largest of these, or 3124213.

(d) All six-digit palindromes are of the form $abccba$, for integers a, b, c, where $1 \leq a \leq 9$ and $0 \leq b, c \leq 9$.
The integer N formed by these digits $abccba$ can be expressed by considering the place value of each of its six digits.
That is, $N = 100000a + 10000b + 1000c + 100c + 10b + a = 100001a + 10010b + 1100c$.
Recognize that the integers $100001, 10010$ and 1100 have a common factor of 11,
so then $N = 11(9091a + 910b + 100c)$ or $N = 11k$ where $k = 9091a + 910b + 100c$.
For N to be divisible by 91, the integer $11k$ must be divisible by 91.
Since 11 is a prime number and not a divisor of 91, then N is divisible by 91 if
$k = 9091a + 910b + 100c$ is divisible by 91. We rewrite k as,

$$\begin{aligned} k &= 9091a + 910b + 100c \\ &= (9100a - 9a) + 910b + (91c + 9c) \\ &= (9100a + 910b + 91c) + (9c - 9a) \\ &= 91(100a + 10b + c) + 9(c - a) \end{aligned}$$

Since $91(100a + 10b + c)$ is divisible by 91 for any choices of a, b, c, then k is divisible by 91 only if $9(c - a)$ is divisible by 91.
Since 9 has no factors in common with 91, $9(c - a)$ is only divisible by 91 if $(c - a)$ is divisible by 91.
Recall that $1 \leq a \leq 9$ and $0 \leq c \leq 9$, thus 91 divides $(c - a)$ only if $c - a = 0$ or $c = a$.
Therefore, the six-digit palindromes that are divisible by 91 are of the form $abaaba$ where $1 \leq a \leq 9$ and $0 \leq b \leq 9$.
Since there are 9 choices for a and 10 choices for b, there are a total of $9 \times 10 = 90$ six-digit palindromes that are divisible by 91.