Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Euclid Contest

Wednesday, April 7, 2010

Solutions

1. (a) *Solution 1*
Since $3^x = 27$, then $3^{x+2} = 3^x 3^2 = 27 \cdot 9 = 243$.

Solution 2
Since $3^x = 27$ and $27 = 3^3$, then $x = 3$.
Therefore, $3^{x+2} = 3^5 = 243$.

(b) Since $2^5 3^{13} 5^9 x = 2^7 3^{14} 5^9$, then $x = \dfrac{2^7 3^{14} 5^9}{2^5 3^{13} 5^9} = 2^2 3^1 = 12$.

(c) The lines $y = x + 2$ and $y = -\frac{1}{2}x + 2$ both pass through the point B on the y-axis.
Since the y-intercept of the line $y = x + 2$ is 2, then B has coordinates $(0, 2)$.
Next, we find the x-intercepts of each of the two lines by setting $y = 0$.
If $y = x + 2$ and $y = 0$, then $x + 2 = 0$ or $x = -2$, so A has coordinates $(-2, 0)$.
If $y = -\frac{1}{2}x + 2$ and $y = 0$, then $0 = -\frac{1}{2}x + 2$ or $\frac{1}{2}x = 2$, and so $x = 4$.
Thus, C has coordinates $(4, 0)$.
Since BO and AC are perpendicular, then we can treat AC as the base of $\triangle ABC$ and BO as its height.
Note that $BO = 2$ and $AC = 4 - (-2) = 6$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2} \times AC \times BO = \frac{1}{2} \times 6 \times 2 = 6$.

2. (a) Let r, g and b be the masses of the red, green and blue packages, respectively.
We are told that $r + g + b = 60$, $r + g = 25$, and $g + b = 50$.
Subtracting the second equation from the first, we obtain $b = 60 - 25 = 35$.
Substituting into the third equation, we obtain $g = 50 - b = 50 - 35 = 15$.
Therefore, the mass of the green package is 15 kg.

(b) Suppose that a palindrome p is the sum of the three consecutive integers $a - 1$, a, $a + 1$.
In this case, $p = (a - 1) + a + (a + 1) = 3a$, so p is a multiple of 3.
The largest palindromes less than 200 are 191, 181, 171.
Note that 191 and 181 are not divisible by 3, but 171 is divisible by 3.
One way to check these without using a calculator is to use the test for divisibility by 3:

> A positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Therefore, 191 and 181 cannot be the sum of three consecutive integers.
The integer 171 can be written as $56 + 57 + 58$, so 171 is the largest palindrome less than 200 that is the sum of three consecutive integers.

(c) *Solution 1*
Since $(x + 1)(x - 1) = 8$, then $x^2 - 1 = 8$ or $x^2 = 9$.
Thus, $(x^2 + x)(x^2 - x) = x(x + 1)x(x - 1) = x^2(x + 1)(x - 1) = 9(8) = 72$.

Solution 2
Since $(x + 1)(x - 1) = 8$, then $x^2 - 1 = 8$ or $x^2 = 9$, so $x = \pm 3$.
If $x = 3$, then $(x^2 + x)(x^2 - x) = (3^2 + 3)(3^2 - 3) = (9 + 3)(9 - 3) = 12(6) = 72$.
If $x = -3$, then $(x^2 + x)(x^2 - x) = ((-3)^2 + (-3))((-3)^2 - (-3)) = (9 - 3)(9 + 3) = 72$.
In either case, $(x^2 + x)(x^2 - x) = 72$.

3. (a) *Solution 1*
Bea spends 60 minutes flying from H to F, 30 minutes at F, 45 minutes flying from F to G, 60 minutes at G, and then flies from G to H.

Thus, her total time is $60 + 30 + 45 + 60 = 195$ minutes plus the length of time that it takes her to fly from G to H.
Since Bea flies at a constant speed, then the ratio of the two distances equals the ratio of the corresponding times.
Therefore, $\dfrac{HF}{GF} = \dfrac{60 \text{ minutes}}{45 \text{ minutes}} = \dfrac{4}{3}$.
Since $\triangle FGH$ is right-angled at F, then $\triangle FGH$ must be similar to a 3-4-5 triangle, and so $\dfrac{HG}{GF} = \dfrac{5}{3}$.
In particular, this means that the ratio of the times flying H to G and F to G is also $\frac{5}{3}$.
Thus, it takes her $\frac{5}{3} \times 45 = 75$ minutes to fly from G to H.
In conclusion, Bea is away from her hive for $195 + 75 = 270$ minutes.

Solution 2
Bea spends 60 minutes flying from H to F, 30 minutes at F, 45 minutes flying from F to G, 60 minutes at G, and then flies from G to H.
Thus, her total time is $60 + 30 + 45 + 60 = 195$ minutes plus the length of time that it takes her to fly from G to H.
Since Bea flies at a constant speed, then the ratio of the two distances equals the ratio of the corresponding times.
Therefore, we can use the Pythagorean Theorem on the *times* to obtain

$$\text{Time } G \text{ to } H = \sqrt{(\text{Time } H \text{ to } F)^2 + (\text{Time } F \text{ to } G)^2} = \sqrt{60^2 + 45^2} = \sqrt{5625} = 75 \text{ min}$$

since the time is positive.
In conclusion, Bea is away from her hive for $195 + 75 = 270$ minutes.

(b) *Solution 1*
Since $\angle OPB = 90^\circ$, then OP and PB are perpendicular, so the product of their slopes is -1.
The slope of OP is $\dfrac{4-0}{p-0} = \dfrac{4}{p}$ and the slope of PB is $\dfrac{4-0}{p-10} = \dfrac{4}{p-10}$.
Therefore, we need

$$\begin{aligned} \frac{4}{p} \cdot \frac{4}{p-10} &= -1 \\ 16 &= -p(p-10) \\ p^2 - 10p + 16 &= 0 \\ (p-2)(p-8) &= 0 \end{aligned}$$

and so $p = 2$ or $p = 8$. Since each these steps is reversible, then $\triangle OPB$ is right-angled precisely when $p = 2$ and $p = 8$.

Solution 2
Since $\triangle OPB$ is right-angled at P, then $OP^2 + PB^2 = OB^2$ by the Pythagorean Theorem.
Note that $OB = 10$ since O has coordinates $(0, 0)$ and B has coordinates $(10, 0)$.
Also, $OP^2 = (p-0)^2 + (4-0)^2 = p^2 + 16$ and $PB^2 = (10-p)^2 + (4-0)^2 = p^2 - 20p + 116$.
Therefore,

$$\begin{aligned} (p^2 + 16) + (p^2 - 20p + 116) &= 10^2 \\ 2p^2 - 20p + 32 &= 0 \\ p^2 - 10p + 16 &= 0 \end{aligned}$$

and so $(p-2)(p-8)=0$, or $p=2$ or $p=8$. Since each these steps is reversible, then $\triangle OPB$ is right-angled precisely when $p=2$ and $p=8$.

4. (a) Suppose that Thurka bought x goats and y helicopters.
Then $19x+17y=201$.
Since x and y are non-negative integers, then $19x \leq 201$ so $x \leq 10$.
If $x=10$, then $17y=201-19x=11$, which does not have an integer solution because 11 is not divisible by 17.
If $x=9$, then $17y=201-19x=30$, which does not have an integer solution.
If $x=8$, then $17y=201-19x=49$, which does not have an integer solution.
If $x=7$, then $17y=201-19x=68$, so $y=4$.
Therefore, $19(7)+17(4)=201$, and so Thurka buys 7 goats and 4 helicopters.
(We can check that $x=0,1,2,3,4,5,6$ do not give values of y that work.)

(b) *Solution 1*
Manipulating algebraically,

$$\begin{aligned}
(x+8)^4 &= (2x+16)^2 \\
(x+8)^4 - 2^2(x+8)^2 &= 0 \\
(x+8)^2((x+8)^2 - 2^2) &= 0 \\
(x+8)^2((x+8)+2)((x+8)-2) &= 0 \\
(x+8)^2(x+10)(x+6) &= 0
\end{aligned}$$

Therefore, $x=-8$ or $x=-10$ or $x=-6$.

Solution 2
Manipulating algebraically,

$$\begin{aligned}
(x+8)^4 &= (2x+16)^2 \\
(x+8)^4 - 2^2(x+8)^2 &= 0 \\
(x+8)^2((x+8)^2 - 2^2) &= 0 \\
(x+8)^2(x^2+16x+64-4) &= 0 \\
(x+8)^2(x^2+16x+60) &= 0 \\
(x+8)^2(x+10)(x+6) &= 0
\end{aligned}$$

Therefore, $x=-8$ or $x=-10$ or $x=-6$.

Solution 3
Since $(x+8)^4=(2x+16)^2$, then $(x+8)^2=2x+16$ or $(x+8)^2=-(2x+16)$.
From the first equation, $x^2+16x+64=2x+16$ or $x^2+14x+48=0$ or $(x+6)(x+8)=0$.
From the second equation, $x^2+16x+64=-2x-16$ or $x^2+18x+80=0$ or $(x+10)(x+8)=0$.
Therefore, $x=-8$ or $x=-10$ or $x=-6$.

5. (a) *Solution 1*

We use the fact that $g(x) = g(f(f^{-1}(x)))$.
Since $f(x) = 2x + 1$, then to determine $f^{-1}(x)$ we solve $x = 2y + 1$ for y to get $2y = x - 1$ or $y = \frac{1}{2}(x - 1)$. Thus, $f^{-1}(x) = \frac{1}{2}(x - 1)$.
Since $g(f(x)) = 4x^2 + 1$, then

$$\begin{aligned} g(x) &= g(f(f^{-1}(x))) \\ &= g(f(\tfrac{1}{2}(x-1))) \\ &= 4(\tfrac{1}{2}(x-1))^2 + 1 \\ &= 4 \cdot \tfrac{1}{4}(x-1)^2 + 1 \\ &= (x-1)^2 + 1 \\ &= x^2 - 2x + 2 \end{aligned}$$

Solution 2

We use the expressions for $f(x)$ and $g(f(x))$ to construct $g(x)$.
Since $f(x)$ is linear and $g(f(x))$ is quadratic, then it is likely that $g(x)$ is also quadratic.
Since $f(x) = 2x + 1$, then $(f(x))^2 = 4x^2 + 4x + 1$.
Since $g(f(x))$ has no term involving x, then we subtract $2f(x)$ (to remove the $4x$ term) to get

$$(f(x))^2 - 2f(x) = (4x^2 + 4x + 1) - 2(2x + 1) = 4x^2 - 1$$

To get $g(f(x))$ from this, we add 2 to get $4x^2 + 1$.
Therefore, $g(f(x)) = (f(x))^2 - 2f(x) + 2$, and so an expression for $g(x)$ is $x^2 - 2x + 2$.

Solution 3

We use the expressions for $f(x)$ and $g(f(x))$ to construct $g(x)$.
Since $f(x)$ is linear and $g(f(x))$ is quadratic, then it is likely that $g(x)$ is also quadratic.
Suppose that $g(x) = ax^2 + bx + c$ for some real numbers a, b, c.
Then

$$\begin{aligned} g(f(x)) &= g(2x+1) \\ &= a(2x+1)^2 + b(2x+1) + c \\ &= a(4x^2 + 4x + 1) + b(2x+1) + c \\ &= 4ax^2 + (4a + 2b)x + (a + b + c) \end{aligned}$$

Since we are told that $g(f(x)) = 4x^2 + 1$, then we can compare coefficients to deduce that $4a = 4$ and $4a + 2b = 0$ and $a + b + c = 1$.
From the first equation, $a = 1$.
From the second equation, $b = -2a = -2$.
From the third equation, $c = 1 - a - b = 2$.
Therefore, an expression for $g(x)$ is $x^2 - 2x + 2$.

(b) *Solution 1*

Since the sum of the first two terms is 40 and the sum of the first three terms is 76, then the third term is $76 - 40 = 36$.
Since the sum of the first three terms is 76 and the sum of the first four terms is 130, then the fourth term is $130 - 76 = 54$.
Since the third term is 36 and the fourth term is 54, then the common ratio in the geometric sequence is $\frac{54}{36} = \frac{3}{2}$.
Therefore, the fifth term is $54 \cdot \frac{3}{2} = 81$ and the sixth term is $81 \cdot \frac{3}{2} = \frac{243}{2}$.

Also, the second term is $36 \div \frac{3}{2} = 36 \cdot \frac{2}{3} = 24$ and the first term is $24 \div \frac{3}{2} = 24 \cdot \frac{2}{3} = 16$.
Thus, the first six terms of the sequence are 16, 24, 36, 54, 81, $\frac{243}{2}$.
Since the first term equals 2^4 and the common ratio is $\frac{3}{2}$, then the nth term in the sequence is $2^4 \left(\frac{3}{2}\right)^{n-1} = \frac{3^{n-1}}{2^{n-5}}$.
When $n \geq 6$, this is a fraction whose numerator is odd and whose denominator is even, and so, when $n \geq 6$, the nth term is not an integer. (An odd integer is never divisible by an even integer.)
Therefore, there will be 5 integers in the sequence.

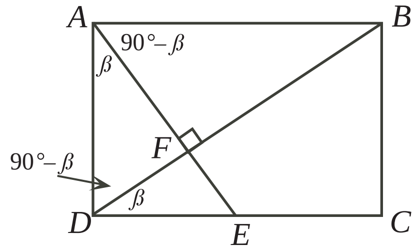
Solution 2
Suppose that a is the first term and r is the common ratio between consecutive terms (so that ar is the second term, ar^2 is the third term, and so on).
From the given information, $a+ar = 40$ and $a+ar+ar^2 = 76$ and $a+ar+ar^2+ar^3 = 130$.
Subtracting the first equation from the second, we obtain $ar^2 = 36$.
Subtracting the second equation from the third, we obtain $ar^3 = 54$.
Since $ar^3 = 54$ and $ar^2 = 36$, then $r = \frac{ar^3}{ar^2} = \frac{54}{36} = \frac{3}{2}$.
Since $ar^2 = 36$ and $r = \frac{3}{2}$, then $a(\frac{3}{2})^2 = 36$ or $\frac{9}{4}a = 36$ or $a = \frac{4}{9} \cdot 36 = 16$.
Since $a = 16$ and $r = \frac{3}{2}$, then the first six terms of the sequence are 16, 24, 36, 54, 81, $\frac{243}{2}$.
Since the first term equals 2^4 and the common ratio is $\frac{3}{2}$, then the nth term in the sequence is $2^4 \left(\frac{3}{2}\right)^{n-1} = \frac{3^{n-1}}{2^{n-5}}$.
When $n \geq 6$, this is a fraction whose numerator is odd and whose denominator is even, and so, when $n \geq 6$, the nth term is not an integer. (An odd integer is never divisible by an even integer.)
Therefore, there will be 5 integers in the sequence.

6. (a) In a 30°-60°-90° triangle, the ratio of the side opposite the 90° to the side opposite the 60° angle is $2 : \sqrt{3}$.
Note that each of $\triangle ABC$, $\triangle ACD$, $\triangle ADE$, $\triangle AEF$, $\triangle AFG$, and $\triangle AGH$ is a 30°-60°-90° triangle.
Therefore, $\frac{AH}{AG} = \frac{AG}{AF} = \frac{AF}{AE} = \frac{AE}{AD} = \frac{AD}{AC} = \frac{AC}{AB} = \frac{2}{\sqrt{3}}$.
Thus, $AH = \frac{2}{\sqrt{3}}AG = \left(\frac{2}{\sqrt{3}}\right)^2 AF = \left(\frac{2}{\sqrt{3}}\right)^3 AE = \left(\frac{2}{\sqrt{3}}\right)^4 AD = \left(\frac{2}{\sqrt{3}}\right)^5 AC = \left(\frac{2}{\sqrt{3}}\right)^6 AB$.
(In other words, to get from $AB = 1$ to the length of AH, we multiply by the "scaling factor" $\frac{2}{\sqrt{3}}$ six times.)
Therefore, $AH = \left(\frac{2}{\sqrt{3}}\right)^6 = \frac{64}{27}$.

(b) *Solution 1*
Since $\triangle AFD$ is right-angled at F, then by the Pythagorean Theorem,

$$AD = \sqrt{AF^2 + FD^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

since $AD > 0$.
Let $\angle FAD = \beta$.
Since $ABCD$ is a rectangle, then $\angle BAF = 90° - \beta$.
Since $\triangle AFD$ is right-angled at F, then $\angle ADF = 90° - \beta$.
Since $ABCD$ is a rectangle, then $\angle BDC = 90° - (90° - \beta) = \beta$.



Therefore, $\triangle BFA$, $\triangle AFD$, and $\triangle DFE$ are all similar as each is right-angled and has either an angle of β or an angle of $90^\circ - \beta$ (and hence both of these angles).

Therefore, $\dfrac{AB}{AF} = \dfrac{DA}{DF}$ and so $AB = \frac{4(2\sqrt{5})}{2} = 4\sqrt{5}$.

Also, $\dfrac{FE}{FD} = \dfrac{FD}{FA}$ and so $FE = \frac{2(2)}{4} = 1$.

Since $ABCD$ is a rectangle, then $BC = AD = 2\sqrt{5}$, and $DC = AB = 4\sqrt{5}$.

Finally, the area of quadrilateral $BCEF$ equals the area of $\triangle DCB$ minus the area $\triangle DFE$.

Thus, the required area is

$$\tfrac{1}{2}(DC)(CB) - \tfrac{1}{2}(DF)(FE) = \tfrac{1}{2}(4\sqrt{5})(2\sqrt{5}) - \tfrac{1}{2}(2)(1) = 20 - 1 = 19$$

Solution 2

Since $\triangle AFD$ is right-angled at F, then by the Pythagorean Theorem,

$$AD = \sqrt{AF^2 + FD^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

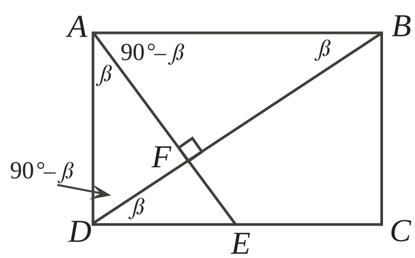
since $AD > 0$.

Let $\angle FAD = \beta$.

Since $ABCD$ is a rectangle, then $\angle BAF = 90^\circ - \beta$. Since $\triangle BAF$ is right-angled at F, then $\angle ABF = \beta$.

Since $\triangle AFD$ is right-angled at F, then $\angle ADF = 90^\circ - \beta$.

Since $ABCD$ is a rectangle, then $\angle BDC = 90^\circ - (90^\circ - \beta) = \beta$.



Looking at $\triangle AFD$, we see that $\sin\beta = \dfrac{FD}{AD} = \dfrac{2}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}}$, $\cos\beta = \dfrac{AF}{AD} = \dfrac{4}{2\sqrt{5}} = \dfrac{2}{\sqrt{5}}$, and $\tan\beta = \dfrac{FD}{AF} = \dfrac{2}{4} = \dfrac{1}{2}$.

Since $AF = 4$ and $\angle ABF = \beta$, then $AB = \dfrac{AF}{\sin\beta} = \dfrac{4}{\frac{1}{\sqrt{5}}} = 4\sqrt{5}$.

Since $FD = 2$ and $\angle FDE = \beta$, then $FE = FD\tan\beta = 2 \cdot \frac{1}{2} = 1$.

Since $ABCD$ is a rectangle, then $BC = AD = 2\sqrt{5}$, and $DC = AB = 4\sqrt{5}$.

Finally, the area of quadrilateral $EFBC$ equals the area of $\triangle DCB$ minus the area $\triangle DFE$.

Thus, the required area is

$$\tfrac{1}{2}(DC)(CB) - \tfrac{1}{2}(DF)(FE) = \tfrac{1}{2}(4\sqrt{5})(2\sqrt{5}) - \tfrac{1}{2}(2)(1) = 20 - 1 = 19$$

7. (a) Using the facts that $9 = 3^2$ and $27 = 3^3$, and the laws for manipulating exponents, we have

$$\begin{aligned} 3^{x-1}9^{\frac{3}{2x^2}} &= 27 \\ 3^{x-1}(3^2)^{\frac{3}{2x^2}} &= 3^3 \\ 3^{x-1}3^{\frac{3}{x^2}} &= 3^3 \\ 3^{x-1+\frac{3}{x^2}} &= 3^3 \end{aligned}$$

When two powers of 3 are equal, their exponents must be equal so

$$\begin{aligned} x - 1 + \frac{3}{x^2} &= 3 \\ x^3 - x^2 + 3 &= 3x^2 \qquad \text{(multiplying by } x^2\text{)} \\ x^3 - 4x^2 + 3 &= 0 \end{aligned}$$

Since $x = 1$ satisfies the equation, then $x - 1$ is a factor of the left side. Using long division or synthetic division, we can factor this out to get $(x-1)(x^2-3x-3) = 0$.
Using the quadratic formula, the quadratic equation $x^2 - 3x - 3 = 0$ has roots

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2} = \frac{3 \pm \sqrt{21}}{2}$$

Therefore, the solutions to the original equation are $x = 1$ and $x = \dfrac{3 \pm \sqrt{21}}{2}$.

(b) To determine the points of intersection, we equate y values of the two curves and obtain $\log_{10}(x^4) = (\log_{10} x)^3$.
Since $\log_{10}(a^b) = b \log_{10} a$, the equation becomes $4 \log_{10} x = (\log_{10} x)^3$.
We set $u = \log_{10} x$ and so the equation becomes $4u = u^3$, or $u^3 - 4u = 0$.
We can factor the left side as $u^3 - 4u = u(u^2 - 4) = u(u+2)(u-2)$.
Therefore, $u(u+2)(u-2) = 0$, and so $u = 0$ or $u = -2$ or $u = 2$.
Therefore, $\log_{10} x = 0$ or $\log_{10} x = -2$ or $\log_{10} x = 2$.
Therefore, $x = 1$ or $x = \frac{1}{100}$ or $x = 100$.
Finally, we must calculate the y-coordinates of the points of intersection. Since one of the original curves is $y = (\log_{10} x)^3$, we can calculate the corresponding values of y by using the fact that $y = u^3$.
The corresponding values of y are $y = 0^3 = 0$ and $y = (-2)^3 = -8$ and $y = 2^3 = 8$.
Therefore, the points of intersection are $(1, 0)$, $(\frac{1}{100}, -8)$ and $(100, 8)$.

8. (a) If Oi-Lam tosses 3 heads, then George has no coins to toss, so cannot toss exactly 1 head.
If Oi-Lam tosses 2, 1 or 0 heads, then George has at least one coin to toss, so can toss exactly 1 head.
Therefore, the following possibilities exist:

* Oi-Lam tosses 2 heads out of 3 coins and George tosses 1 head out of 1 coin
* Oi-Lam tosses 1 head out of 3 coins and George tosses 1 head out of 2 coins
* Oi-Lam tosses 0 heads out of 3 coins and George tosses 1 head out of 3 coins

We calculate the various probabilities.
If 3 coins are tossed, there are 8 equally likely possibilities: HHH, HHT, HTH, THH, TTH, THT, HTT, TTT. Each of these possibilities has probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Therefore,

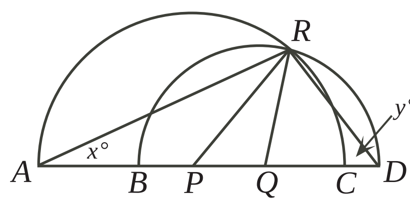
* the probability of tossing 0 heads out of 3 coins is $\frac{1}{8}$
* the probability of tossing 1 head out of 3 coins is $\frac{3}{8}$
* the probability of tossing 2 heads out of 3 coins is $\frac{3}{8}$
* the probability of tossing 3 heads out of 3 coins is $\frac{1}{8}$

If 2 coins are tossed, there are 4 equally likely possibilities: HH, HT, TH, TT. Each of these possibilities has probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore, the probability of tossing 1 head out of 2 coins is $\frac{2}{4} = \frac{1}{2}$.
If 1 coin is tossed, the probability of tossing 1 head is $\frac{1}{2}$.
To summarize, the possibilities are

* Oi-Lam tosses 2 heads out of 3 coins (with probability $\frac{3}{8}$) and George tosses 1 head out of 1 coin (with probability $\frac{1}{2}$)
* Oi-Lam tosses 1 head out of 3 coins (with probability $\frac{3}{8}$) and George tosses 1 head out of 2 coins (with probability $\frac{1}{2}$)
* Oi-Lam tosses 0 heads out of 3 coins (with probability $\frac{1}{8}$) and George tosses 1 head out of 3 coins (with probability $\frac{3}{8}$)

Therefore, the overall probability is $\frac{3}{8} \cdot \frac{1}{2} + \frac{3}{8} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{3}{8} = \frac{27}{64}$.

(b) Suppose $\angle PAR = x^\circ$ and $\angle QDR = y^\circ$.



Since PR and PA are radii of the larger circle, then $\triangle PAR$ is isosceles.
Thus, $\angle PRA = \angle PAR = x^\circ$.
Since QD and QR are radii of the smaller circle, then $\triangle QRD$ is isosceles.
Thus, $\angle QRD = \angle QDR = y^\circ$.
In $\triangle ARD$, the sum of the angles is 180°, so $x^\circ + (x^\circ + 40^\circ + y^\circ) + y^\circ = 180^\circ$ or $2x + 2y = 140$ or $x + y = 70$.
Therefore, $\angle ARD = x^\circ + 40^\circ + y^\circ = (x + y + 40)^\circ = 110^\circ$.

9. (a) (i) *Solution 1*

$$\begin{aligned}
\text{LS} &= \cot\theta - \cot 2\theta \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta} \\
&= \frac{\sin 2\theta \cos\theta - \cos 2\theta \sin\theta}{\sin\theta \sin 2\theta} \\
&= \frac{\sin(2\theta - \theta)}{\sin\theta \sin 2\theta} \\
&= \frac{\sin\theta}{\sin\theta \sin 2\theta} \\
&= \frac{1}{\sin 2\theta} \\
&= \text{RS}
\end{aligned}$$

as required.

Solution 2

$$\begin{aligned}
\text{LS} &= \cot\theta - \cot 2\theta \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta} \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{2\sin\theta\cos\theta} \\
&= \frac{2\cos^2\theta - \cos 2\theta}{2\sin\theta\cos\theta} \\
&= \frac{2\cos^2\theta - (2\cos^2\theta - 1)}{\sin 2\theta} \\
&= \frac{1}{\sin 2\theta} \\
&= \text{RS}
\end{aligned}$$

as required.

(ii) We use (i) to note that $\frac{1}{\sin 8^\circ} = \cot 4^\circ - \cot 8^\circ$ and $\frac{1}{\sin 16^\circ} = \cot 8^\circ - \cot 16^\circ$ and so on. Thus,

$$\begin{aligned}
S &= \frac{1}{\sin 8^\circ} + \frac{1}{\sin 16^\circ} + \frac{1}{\sin 32^\circ} + \cdots + \frac{1}{\sin 4096^\circ} + \frac{1}{\sin 8192^\circ} \\
&= (\cot 4^\circ - \cot 8^\circ) + (\cot 8^\circ - \cot 16^\circ) + (\cot 16^\circ - \cot 32^\circ) + \\
&\qquad \cdots + (\cot 2048^\circ - \cot 4096^\circ) + (\cot 4096^\circ - \cot 8192^\circ) \\
&= \cot 4^\circ - \cot 8192^\circ
\end{aligned}$$

since the sum “telescopes”.
Since the cotangent function has a period of 180°, and 8100° is a multiple of 180°, then $\cot 8192^\circ = \cot 92^\circ$.
Therefore,

$$\begin{aligned}
S &= \cot 4^\circ - \cot 92^\circ \\
&= \frac{\cos 4^\circ}{\sin 4^\circ} - \frac{\cos 92^\circ}{\sin 92^\circ} \\
&= \frac{\cos 4^\circ}{\sin 4^\circ} - \frac{-\sin 2^\circ}{\cos 2^\circ} \\
&= \frac{\cos 4^\circ}{2\sin 2^\circ\cos 2^\circ} + \frac{\sin 2^\circ}{\cos 2^\circ} \\
&= \frac{\cos 4^\circ + 2\sin^2 2^\circ}{2\sin 2^\circ\cos 2^\circ} \\
&= \frac{(1 - 2\sin^2 2^\circ) + 2\sin^2 2^\circ}{\sin 4^\circ} \\
&= \frac{1}{\sin 4^\circ}
\end{aligned}$$

Therefore, $\alpha = 4^\circ$.

(b) *Solution 1*

We use the notation $A = \angle BAC$, $B = \angle ABC$ and $C = \angle ACB$.

We need to show that $A < \frac{1}{2}(B + C)$. Since the sum of the angles in $\triangle ABC$ is 180°, then $B + C = 180^\circ - A$, and so this inequality is equivalent to $A < \frac{1}{2}(180^\circ - A)$ which is equivalent to $\frac{3}{2}A < 90^\circ$ or $A < 60^\circ$.

So we need to show that $A < 60^\circ$.

We know that $a < \frac{1}{2}(b + c)$. Thus, $2a < b + c$ and so $4a^2 < b^2 + c^2 + 2bc$ because all quantities are positive.

Using the cosine law in $\triangle ABC$, we obtain $a^2 = b^2 + c^2 - 2bc \cos A$.

Therefore,

$$\begin{aligned}
4a^2 &< b^2 + c^2 + 2bc \\
4(b^2 + c^2 - 2bc \cos A) &< b^2 + c^2 + 2bc \\
4b^2 + 4c^2 - 8bc \cos A &< b^2 + c^2 + 2bc \\
4b^2 + 4c^2 - 8bc \cos A &< b^2 + c^2 + 2bc + 3(b - c)^2 \qquad (\text{since } (b - c)^2 \geq 0) \\
4b^2 + 4c^2 - 8bc \cos A &< b^2 + c^2 + 2bc + 3b^2 - 6bc + 3c^2 \\
4b^2 + 4c^2 - 8bc \cos A &< 4b^2 + 4c^2 - 4bc \\
-8bc \cos A &< -4bc \\
\cos A &> \tfrac{1}{2} \qquad (\text{since } 8bc > 0)
\end{aligned}$$

Since $2a < b + c$, then a cannot be the longest side of $\triangle ABC$ (that is, we cannot have $a \geq b$ and $a \geq c$), so A must be an acute angle.

Therefore, $\cos A > \frac{1}{2}$ implies $A < 60^\circ$, as required.

Solution 2

We use the notation $A = \angle BAC$, $B = \angle ABC$ and $C = \angle ACB$.

We need to show that $A < \frac{1}{2}(B + C)$. Since the sum of the angles in $\triangle ABC$ is 180°, then $B + C = 180^\circ - A$, and so this inequality is equivalent to $A < \frac{1}{2}(180^\circ - A)$ which is equivalent to $\frac{3}{2}A < 90^\circ$ or $A < 60^\circ$.

So we need to show that $A < 60^\circ$.

We know that $a < \frac{1}{2}(b + c)$ which implies $2a < b + c$.

Using the sine law in $\triangle ABC$, we obtain $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, which gives $b = \dfrac{a \sin B}{\sin A}$ and $c = \dfrac{a \sin C}{\sin A}$.

Therefore, we obtain equivalent inequalities

$$\begin{aligned}
2a &< b + c \\
2a &< \frac{a \sin B}{\sin A} + \frac{a \sin C}{\sin A} \\
2a \sin A &< a \sin B + a \sin C \qquad (\text{since } \sin A > 0 \text{ for } 0^\circ < A < 180^\circ) \\
2 \sin A &< \sin B + \sin C
\end{aligned}$$

since $a > 0$. Next, we use the trigonometric formula $\sin B + \sin C = 2 \sin\left(\dfrac{B + C}{2}\right) \cos\left(\dfrac{B - C}{2}\right)$.

Since $\cos\theta \leq 1$ for any θ, then $\sin B + \sin C \leq 2 \sin\left(\dfrac{B + C}{2}\right) \cdot 1 = 2 \sin\left(\dfrac{B + C}{2}\right)$.

Therefore,

$$\begin{aligned} 2\sin A &< \sin B + \sin C \leq 2\sin\left(\frac{B+C}{2}\right) \\ 2\sin A &< 2\sin\left(\frac{B+C}{2}\right) \\ 2\sin A &< 2\sin\left(\frac{180^\circ - A}{2}\right) \\ 4\sin(\tfrac{1}{2}A)\cos(\tfrac{1}{2}A) &< 2\sin(90^\circ - \tfrac{1}{2}A) \\ 2\sin(\tfrac{1}{2}A)\cos(\tfrac{1}{2}A) &< \cos(\tfrac{1}{2}A) \end{aligned}$$

Since $0^\circ < A < 180^\circ$, then $\cos(\frac{1}{2}A) > 0$, so $\sin(\frac{1}{2}A) < \frac{1}{2}$.
Since $2a < b + c$, then a cannot be the longest side of $\triangle ABC$, so A must be an acute angle.
Therefore, $\frac{1}{2}A < 30^\circ$ or $A < 60^\circ$, as required.

10. Denote the side lengths of a triangle by a, b and c, with $0 < a \leq b \leq c$.
In order for these lengths to form a triangle, we need $c < a + b$ and $b < a + c$ and $a < b + c$.
Since $0 < a \leq b \leq c$, then $b < a + c$ and $a < b + c$ follow automatically, so only $c < a + b$ ever needs to be checked.
Instead of directly considering triangles and sets of triangle, we can consider triples (a, b, c) and sets of triples (a, b, c) with the appropriate conditions.
For each positive integer $k \geq 3$, we use the notation S_k to denote the set of triples of positive integers (a, b, c) with $0 < a \leq b \leq c$ and $c < a + b$ and $a + b + c = k$.
In this case, $c < a + b$ and $a + b + c = k$, so $c + c < a + b + c = k$, so $2c < k$ or $c < \frac{1}{2}k$.
Also, if $0 < a \leq b \leq c$ and $a + b + c = k$, then $k = a + b + c \leq c + c + c$, so $3c \geq k$ or $c \geq \frac{1}{3}k$.

(a) Consider $T(10)$, which is the number of elements in S_{10}.
We want to find all possible triples (a, b, c) of integers with $0 < a \leq b \leq c$ and $c < a + b$ and $a + b + c = 10$.
We need $c < \frac{10}{2} = 5$ and $c \geq \frac{10}{3}$. Thus, $c = 4$.
Therefore, we need $0 < a \leq b \leq 4$ and $a + b = 6$.
There are two possibilities: $(a, b, c) = (2, 4, 4)$ or $(a, b, c) = (3, 3, 4)$.
Therefore, $T(10) = 2$.

Consider $T(11)$. We want to find all possible triples (a, b, c) of integers with $0 < a \leq b \leq c$ and $c < a + b$ and $a + b + c = 11$.
We need $c < \frac{11}{2}$ and $c \geq \frac{11}{3}$. Thus, $c = 4$ or $c = 5$.
If $c = 4$, we need $0 < a \leq b \leq 4$ and $a + b = 7$.
There is only one possibility: $(a, b, c) = (3, 4, 4)$.
If $c = 5$, we need $0 < a \leq b \leq 5$ and $a + b = 6$.
There are three possibilities: $(a, b, c) = (1, 5, 5)$ or $(a, b, c) = (2, 4, 5)$ or $(a, b, c) = (3, 3, 5)$.
Therefore, $T(11) = 4$.

Consider $T(12)$. We want to find all possible triples (a, b, c) of integers with $0 < a \leq b \leq c$ and $c < a + b$ and $a + b + c = 12$.
We need $c < \frac{12}{2}$ and $c \geq \frac{12}{3}$. Thus, $c = 4$ or $c = 5$.
If $c = 4$, we need $0 < a \leq b \leq 4$ and $a + b = 8$.
There is only one possibility: $(a, b, c) = (4, 4, 4)$.

If $c = 5$, we need $0 < a \leq b \leq 5$ and $a + b = 7$.
There are two possibilities: $(a, b, c) = (2, 5, 5)$ or $(a, b, c) = (3, 4, 5)$.
Therefore, $T(12) = 3$.

(b) We show that $T(2m) = T(2m - 3)$ by creating a one-to-one correspondence between the triples in S_{2m} and the triples S_{2m-3}.
Note that S_{2m} is the set of triples (a, b, c) of positive integers with $0 < a \leq b \leq c$, with $c < a + b$, and with $a + b + c = 2m$.
Also, S_{2m-3} is the set of triples (A, B, C) of positive integers with $0 < A \leq B \leq C$, with $C < A + B$, and with $A + B + C = 2m - 3$.

Consider a triple (a, b, c) in S_{2m} and a corresponding triple $(a - 1, b - 1, c - 1)$.
We show that $(a - 1, b - 1, c - 1)$ is in S_{2m-3}:

* Since (a, b, c) is in S_{2m}, then $c < \frac{1}{2}(2m) = m$. This means that $b \leq c \leq m - 1$, so $a = 2m - b - c \geq 2$. Therefore, $a - 1$, $b - 1$ and $c - 1$ are positive integers since a, b and c are positive integers with $2 \leq a \leq b \leq c$.
* Since $2 \leq a \leq b \leq c$, then $1 \leq a - 1 \leq b - 1 \leq c - 1$, so $0 < a - 1 \leq b - 1 \leq c - 1$.
* Since $a + b + c = 2m$, then $c = 2m - (a + b)$ so $a + b$ and c have the same parity.
  Since $c < a + b$, then $c \leq a + b - 2$. (In other words, it cannot be the case that $c = a + b - 1$.) Therefore, $c - 1 \leq (a - 1) + (b - 1) - 1$; that is, $c - 1 < (a - 1) + (b - 1)$.
* Since $a + b + c = 2m$, then $(a - 1) + (b - 1) + (c - 1) = 2m - 3$.

Therefore, $(a - 1, b - 1, c - 1)$ is in S_{2m-3}, since it satisfies all of the conditions of S_{2m-3}.
Note as well that two different triples in S_{2m} correspond to two different triples in S_{2m-3}.
Thus, every triple in S_{2m} corresponds to a different triple in S_{2m-3}.
Thus, $T(2m) \leq T(2m - 3)$.

Consider a triple (A, B, C) in S_{2m-3} and a corresponding triple $(A + 1, B + 1, C + 1)$.
We show that $(A + 1, B + 1, C + 1)$ is in S_{2m}:

* Since (A, B, C) is in S_{2m-3}, then A, B and C are positive integers, so $A + 1$, $B + 1$ and $C + 1$ are positive integers.
* Since $0 < A \leq B \leq C$, then $1 < A+1 \leq B+1 \leq C+1$, so $0 < A+1 \leq B+1 \leq C+1$.
* Since $C < A + B$, then $C + 1 < (A + 1) + (B + 1) - 1$ so $C + 1 < (A + 1) + (B + 1)$.
* Since $A + B + C = 2m - 3$, then $(A + 1) + (B + 1) + (C + 1) = 2m$.

Therefore, $(A + 1, B + 1, C + 1)$ is in S_{2m}.
Note again that two different triples in S_{2m-3} correspond to two different triples in S_{2m}.
Thus, every triple in S_{2m-3} corresponds to a different triple in S_{2m}.
Therefore, $T(2m - 3) \leq T(2m)$.

Since $T(2m) \leq T(2m - 3)$ and $T(2m - 3) \leq T(2m)$, then $T(2m) = T(2m - 3)$.

(c) We will use two important facts:

(F1) $T(2m) = T(2m - 3)$ for every positive integer $m \geq 3$, and

(F2) $T(k) \leq T(k + 2)$ for every positive integer $k \geq 3$

We proved (F1) in (b).
Next, we prove (F2):

Consider a triple (a, b, c) in S_k and a corresponding triple $(a, b + 1, c + 1)$. We show that the triple $(a, b + 1, c + 1)$ is in S_{k+2}:

* Since a, b and c are positive integers, then a, $b + 1$ and $c + 1$ are positive integers.

* Since $0 < a \leq b \leq c$, then $0 < a \leq b+1 \leq c+1$.
* Since $c < a+b$, then $c+1 < a+(b+1)$.
* Since $a+b+c=k$, then $a+(b+1)+(c+1)=k+2$.

Therefore, $(a, b+1, c+1)$ is in S_{k+2}. Note that, using this correspondence, different triples in S_k correspond different triples in S_{k+2}. Thus, every triple in S_k corresponds to a different triple in S_{k+2}. This proves that $T(k) \leq T(k+2)$.

Suppose that $n = N$ is the smallest positive integer for which $T(n) > 2010$.
Then N must be odd:

If N was even, then by (F1), $T(N-3) = T(N) > 2010$ and so $n = N-3$ would be an integer smaller than N with $T(n) > 2010$. This contradicts the fact that $n = N$ is the smallest such integer.

Therefore, we want to find the smallest odd positive integer N for which $T(N) > 2010$.
Next, we note that if we can find an odd positive integer n such that $T(n) > 2010 \geq T(n-2)$, then we will have found the desired value of n:

This is because n and $n-2$ are both odd, and by property (F2), any smaller odd positive integer k will give $T(k) \leq T(n-2) \leq 2010$ and any larger odd positive integer m will give $T(m) \geq T(n) > 2010$.

We show that $N = 309$ is the desired value of N by showing that $T(309) > 2010$ and $T(307) \leq 2010$.

Calculation of $T(309)$
We know that $\frac{309}{3} \leq c < \frac{309}{2}$, so $103 \leq c \leq 154$.
For each admissible value of c, we need to count the number of pairs of positive integers (a, b) with $a \leq b \leq c$ and $a+b = 309-c$.
For example, if $c = 154$, then we need $a \leq b \leq 154$ and $a+b = 155$.
This gives pairs $(1,154), (2,153), \ldots, (76,79), (77,78)$, of which there are 77.
Also, if $c = 153$, then we need $a \leq b \leq 153$ and $a+b = 156$.
This gives pairs $(3,153), \ldots, (77,79), (78,78)$, of which there are 76.
In general, if c is even, then the minimum possible value of a occurs when b is as large as possible – that is, when $b = c$, so $a \geq 309-2c$.
Also, the largest possible value of a occurs when a and b are as close to equal as possible.
Since c is even, then $309-c$ is odd, so a and b cannot be equal, but they can differ by 1.
In this case, $a = 154 - \frac{1}{2}c$ and $b = 155 - \frac{1}{2}c$.
Therefore, if c is even, there are $(154-\frac{1}{2}c) - (309-2c) + 1 = \frac{3}{2}c - 154$ possible pairs (a, b) and so $\frac{3}{2}c - 154$ possible triples.
In general, if c is odd, then the minimum possible value of a occurs when b is as large as possible – that is, when $b = c$, so $a \geq 309-2c$.
Also, the largest possible value of a occurs when a and b are as close to equal as possible.
Since c is odd, then $309-c$ is even, so a and b can be equal. In this case, $a = \frac{1}{2}(309-c)$.
Therefore, if c is odd, there are $\frac{1}{2}(309-c) - (309-2c) + 1 = \frac{3}{2}c - \frac{307}{2}$ possible pairs (a, b) and so $\frac{3}{2}c - \frac{307}{2}$ possible triples.
The possible even values of c are $104, 106, \ldots, 152, 154$ (there are 26 such values) and the possible odd values of c are $103, 105, \ldots, 151, 153$ (there are 26 such values).

Therefore,

$$\begin{aligned}
T(309) &= \left(\tfrac{3}{2}(104)-154\right)+\left(\tfrac{3}{2}(106)-154\right)+\cdots+\left(\tfrac{3}{2}(154)-154\right)+ \\
&\qquad \left(\tfrac{3}{2}(103)-\tfrac{307}{2}\right)+\left(\tfrac{3}{2}(105)-\tfrac{307}{2}\right)+\cdots+\left(\tfrac{3}{2}(153)-\tfrac{307}{2}\right) \\
&= \tfrac{3}{2}(104+106+\cdots+154)-26\cdot 154+\tfrac{3}{2}(103+105+\cdots+153)-26\cdot\tfrac{307}{2} \\
&= \tfrac{3}{2}(103+104+105+106+\cdots+153+154)-26\cdot 154-26\cdot\tfrac{307}{2} \\
&= \tfrac{3}{2}\cdot\tfrac{1}{2}(103+154)(52)-26\cdot 154-26\cdot\tfrac{307}{2} \\
&= \tfrac{3}{2}(26)(257)-26\cdot 154-26\cdot\tfrac{307}{2} \\
&= 2028
\end{aligned}$$

Therefore, $T(309)>2010$, as required.

Calculation of $T(307)$

We know that $\frac{307}{3}\le c<\frac{307}{2}$, so $103\le c\le 153$.
For each admissible value of c, we need to count the number of pairs of positive integers (a,b) with $a\le b\le c$ and $a+b=307-c$.
This can be done in a similar way to the calculation of $T(309)$ above.
If n is even, there are $\frac{3}{2}c-153$ possible triples.
If n is odd, there are $\frac{3}{2}c-\frac{305}{2}$ possible triples.
The possible even values of c are $104,106,\ldots,150,152$ (there are 25 such values) and the possible odd values of c are $103,105,\ldots,151,153$ (there are 26 such values).
Therefore,

$$\begin{aligned}
T(307) &= \left(\tfrac{3}{2}(104)-153\right)+\left(\tfrac{3}{2}(106)-153\right)+\cdots+\left(\tfrac{3}{2}(152)-153\right)+ \\
&\qquad \left(\tfrac{3}{2}(103)-\tfrac{305}{2}\right)+\left(\tfrac{3}{2}(105)-\tfrac{305}{2}\right)+\cdots+\left(\tfrac{3}{2}(153)-\tfrac{305}{2}\right) \\
&= \tfrac{3}{2}(104+106+\cdots+152)-25\cdot 153+\tfrac{3}{2}(103+105+\cdots+153)-26\cdot\tfrac{305}{2} \\
&= \tfrac{3}{2}(103+104+105+106+\cdots+152+153)-25\cdot 153-26\cdot\tfrac{305}{2} \\
&= \tfrac{3}{2}\cdot\tfrac{1}{2}(103+153)(51)-25\cdot 153-26\cdot\tfrac{305}{2} \\
&= \tfrac{3}{2}(51)(128)-25\cdot 153-26\cdot\tfrac{305}{2} \\
&= 2002
\end{aligned}$$

Therefore, $T(307)<2010$, as required.

Therefore, the smallest positive integer n such that $T(n)>2010$ is $n=309$.

As a final note, we discuss briefly how one could guess that the answer was near $N=309$.

Consider the values of $T(n)$ for small odd positive integers n.
In (a), by considering the possible values of c from smallest (roughly $\frac{1}{3}n$) to largest (roughly $\frac{1}{2}n$), we saw that $T(11)=1+3=4$.
If we continue to calculate $T(n)$ for a few more small odd values of n we will see that:

$$\begin{aligned}
T(13) &= 2+3=5 \\
T(15) &= 1+2+4=7 \\
T(17) &= 1+3+4=8 \\
T(19) &= 2+3+5=10 \\
T(21) &= 1+2+4+5=12 \\
T(23) &= 1+3+4+6=14
\end{aligned}$$

The pattern that seems to emerge is that for n odd, $T(n)$ is roughly equal to the sum of the integers from 1 to $\frac{1}{4}n$, with one out of every three integers removed.
Thus, $T(n)$ is roughly equal to $\frac{2}{3}$ of the sum of the integers from 1 to $\frac{1}{4}n$.
Therefore, $T(n) \approx \frac{2}{3} \cdot \frac{1}{2}(\frac{1}{4}n)(\frac{1}{4}n+1) \approx \frac{2}{3} \cdot \frac{1}{2}(\frac{1}{4}n)^2 \approx \frac{1}{48}n^2$.
It makes sense to look for an odd positive integer n with $T(n) \approx 2010$.
Thus, we are looking for a value of n that roughly satisfies $\frac{1}{48}n^2 \approx 2010$ or $n^2 \approx 96480$ or $n \approx 310$.
Since n is odd, then it makes sense to consider $n = 309$, as in the solution above.