Canadian
Mathematics
Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Cayley Contest

(Grade 10)

Thursday, February 25, 2010

Solutions

1. Calculating, $6 + 4 \div 2 = 6 + 2 = 8$.

   Answer: (D)

2. Since there are 12 equally spaced numbers and the total angle in a complete circle is 360°, then the angle between two consecutive numbers is $360^\circ \div 12 = 30^\circ$.
   To rotate 120°, the minute hand must move by $120^\circ \div 30^\circ = 4$ numbers clockwise from the 12.
   Therefore, the hand will be pointing at the 4.

   Answer: (C)

3. Since $x + \sqrt{25} = \sqrt{36}$, then $x + 5 = 6$ or $x = 1$.

   Answer: (A)

4. Evaluating, $\dfrac{1}{2 + \frac{2}{3}} = \dfrac{1}{\frac{8}{3}} = \dfrac{3}{8}$.

   Answer: (E)

5. In a rectangle, length times width equals area, so width equals area divided by length.
   Therefore, the width is $\frac{1}{3} \div \frac{3}{5} = \frac{1}{3} \times \frac{5}{3} = \frac{5}{9}$.

   Answer: (B)

6. Since the sum of the angles in a triangle is 180°, then $3x^\circ + x^\circ + 6x^\circ = 180^\circ$ or $10x = 180$ or $x = 18$.
   The largest angle in the triangle is $6x^\circ = 6(18^\circ) = 108^\circ$.

   Answer: (E)

7. *Solution 1*
   Since the mean of five consecutive integers is 9, then the middle of these five integers is 9.
   Therefore, the integers are $7, 8, 9, 10, 11$, and so the smallest of the five integers is 7.

   Solution 2
   Suppose that x is the smallest of the five consecutive integers.
   Then the integers are x, $x + 1$, $x + 2$, $x + 3$, and $x + 4$.
   The mean of these integers is $\dfrac{x + (x + 1) + (x + 2) + (x + 3) + (x + 4)}{5} = \dfrac{5x + 10}{5} = x + 2$.
   Since the mean is 9, then $x + 2 = 9$ or $x = 7$.
   Thus, the smallest of the five integers is 7.

   Answer: (D)

8. Since square $PQRS$ has an area of 900, then its side length is $\sqrt{900} = 30$.
   Thus, $PQ = PS = 30$.
   Since M and N are the midpoints of PQ and PS, respectively, then $PN = PM = \frac{1}{2}(30) = 15$.
   Since $PQRS$ is a square, then the angle at P is 90°, so $\triangle PMN$ is right-angled.
   Therefore, the area of $\triangle PMN$ is $\frac{1}{2}(PM)(PN) = \frac{1}{2}(15)(15) = \frac{225}{2} = 112.5$.

   An alternate approach would be to divide the square into 8 triangles, each congruent to $\triangle PMN$. (Can you see how to do this?) This would tell us that the area of $\triangle PMN$ is $\frac{1}{8}$ of the area of the square, or $\frac{1}{8}(30^2) = 112.5$.

   Answer: (B)

9. *Solution 1*

Since the triangle will include the two axes, then the triangle will have a right angle.
For the triangle to be isosceles, the other two angles must be 45°.
For a line to make an angle of 45° with both axes, it must have slope 1 or -1.
Of the given possibilities, the only such line is $y = -x + 4$.

Solution 2

The vertices of a triangle formed in this way will be the points of intersection of the pairs of lines forming the triangle.
The point of intersection of the x- and y-axes is $(0, 0)$.
The line $y = -x + 4$ crosses the y-axis at $(0, 4)$.
The line $y = -x + 4$ crosses the x-axis where $y = 0$, or $-x + 4 = 0$ and so at the point $(4, 0)$.
Therefore, if the third line is $y = -x+4$, the vertices of the triangle are $(0, 0)$, $(0, 4)$ and $(4, 0)$.
This triangle is isosceles since two of its sides have length 4.
Therefore, the third side is $y = -x + 4$, since we know that only one possibility can be correct.

Answer: (C)

10. Since the ratio of boys to girls at Pascal H.S. is $3 : 2$, then $\frac{3}{3+2} = \frac{3}{5}$ of the students at Pascal H.S. are boys.
Thus, there are $\frac{3}{5}(400) = \frac{1200}{5} = 240$ boys at Pascal H.S.
Since the ratio of boys to girls at Fermat C.I. is $2 : 3$, then $\frac{2}{2+3} = \frac{2}{5}$ of the students at Fermat C.I. are boys.
Thus, there are $\frac{2}{5}(600) = \frac{1200}{5} = 240$ boys at Fermat C.I.
There are $400 + 600 = 1000$ students in total at the two schools.
Of these, $240 + 240 = 480$ are boys, and so the remaining $1000 - 480 = 520$ students are girls.
Therefore, the overall ratio of boys to girls is $480 : 520 = 48 : 52 = 12 : 13$.

Answer: (B)

11. First, we note that the values of x and y cannot be equal since they are integers and $x + y$ is odd.
Next, we look at the case when $x > y$.
We list the fifteen possible pairs of values for x and y and the corresponding values of xy:

x	y	xy	x	y	xy	x	y	xy
30	1	30	25	6	150	20	11	220
29	2	58	24	7	168	19	12	228
28	3	84	23	8	184	18	13	234
27	4	108	22	9	198	17	14	238
26	5	130	21	10	210	16	15	240

Therefore, the largest possible value for xy is 240.
Note that the largest value occurs when x and y are as close together as possible.
The case of $x < y$ will give us exactly the same result. We can see this by switching the headings of x and y in the table above.

Answer: (A)

12. *Solution 1*

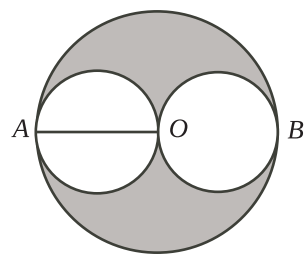
Since the sale price has been reduced by 20%, then the sale price of \$112 is 80% or $\frac{4}{5}$ of the regular price.

Therefore, $\frac{1}{5}$ of the regular price is $\$112 \div 4 = \$28$.
Thus, the regular price is $\$28 \times 5 = \$140$.
If the regular price is reduced by 30%, the new sale price would be 70% of the regular price, or $\frac{7}{10}(\$140) = \$98$.

Solution 2
Suppose that the original price was $\$x$.
Since the price has been reduced by 20%, then the sale price is 80% of the original price, or $\frac{80}{100}x = \frac{8}{10}x$.
Therefore, $\frac{8}{10}x = 112$.
If it were on sale for 30% off, then the price would be 70% of the original price, or $\frac{7}{10}x$.
Now, $\frac{7}{10}x = \frac{7}{8}\left(\frac{8}{10}x\right) = \frac{7}{8}(112) = 98$.
Thus, the price would be \$98.

Answer: (E)

13. Label the centre of the larger circle O and the points of contact between the larger circle and the smaller circles A and B. Draw the radius OA of the larger circle.



Since the smaller circle and the larger circle touch at A, then the diameter through A of the smaller circle lies along the diameter through A of the larger circle. (This is because each diameter is perpendicular to the common tangent at the point of contact.)
Since AO is a radius of the larger circle, then it is a diameter of the smaller circle.
Since the radius of the larger circle is 6, then the diameter of the smaller circle is 6, so the radius of the smaller circle on the left is 3.
Similarly, we can draw a radius through O and B and deduce that the radius of the smaller circle on the right is also 3.
The area of the shaded region equals the area of the larger circle minus the combined area of the two smaller circles.
Thus, the area of the shaded region is $6^2\pi - 3^2\pi - 3^2\pi = 36\pi - 9\pi - 9\pi = 18\pi$.

Answer: (D)

14. Since b is a positive integer, then $b^2 \geq 1$, and so $a^2 \leq 49$, which gives $1 \leq a \leq 7$, since a is a positive integer.
If $a = 7$, then $b^2 = 50 - 7^2 = 1$, so $b = 1$.
If $a = 6$, then $b^2 = 50 - 6^2 = 14$, which is not possible since b is an integer.
If $a = 5$, then $b^2 = 50 - 5^2 = 25$, so $b = 5$.
If $a = 4$, then $b^2 = 50 - 4^2 = 34$, which is not possible.
If $a = 3$, then $b^2 = 50 - 3^2 = 41$, which is not possible.
If $a = 2$, then $b^2 = 50 - 2^2 = 46$, which is not possible.
If $a = 1$, then $b^2 = 50 - 1^2 = 49$, so $b = 7$.
Therefore, there are 3 pairs (a, b) that satisfy the equation, namely $(7, 1), (5, 5), (1, 7)$.

Answer: (C)

15. Since the coins in the bag of loonies are worth \$400, then there are 400 coins in the bag.
Since 1 loonie has the same mass as 4 dimes, then 400 loonies have the same mass as 4(400) or 1600 dimes.
Therefore, the bag of dimes contains 1600 dimes, and so the coins in this bag are worth \$160.

ANSWER: (C)

16. The sum of the odd numbers from 5 to 21 is

$$5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 = 117$$

Therefore, the sum of the numbers in any row is one-third of this total, or 39.
This means as well that the sum of the numbers in any column or diagonal is also 39.
Since the numbers in the middle row add to 39, then the number in the centre square is $39 - 9 - 17 = 13$.
Since the numbers in the middle column add to 39, then the number in the middle square in the bottom row is $39 - 5 - 13 = 21$.

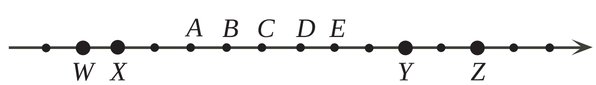
	5	
9	13	17
x	21	

Since the numbers in the bottom row add to 39, then the number in the bottom right square is $39 - 21 - x = 18 - x$.
Since the numbers in the bottom left to top right diagonal add to 39, then the number in the top right square is $39 - 13 - x = 26 - x$.
Since the numbers in the rightmost column add to 39, then $(26 - x) + 17 + (18 - x) = 39$ or $61 - 2x = 39$ or $2x = 22$, and so $x = 11$.
We can complete the magic square as follows:

19	5	15
9	13	17
11	21	7

ANSWER: (B)

17. We label the points corresponding to the four larger dots as W, X, Y, and Z.



Two of these four points must be multiples of 5.
These multiples of 5 must differ from each other by a multiple of 5.
We look at the possible differences between the four numbers W, X, Y, and Z:

$$X - W = 1,\ Y - W = 9,\ Y - X = 8,\ Z - W = 11,\ Z - X = 10,\ Z - Y = 2$$

Only X and Z differ by a multiple of 5. Thus, X and Z must be the two multiples of 5.
We know that one of A, B, C, D, E is a multiple of 5, and so it must differ from X by a multiple of 5.
Since $D - X = 5$, then D is the only multiple of 5 among A, B, C, D, E (the others differ from X by 2, 3, 4, and 6).
Since D is the only possibility that is a multiple of 5, then D must be the multiple of 15.
(We can check that X and Z are multiples of 5 and W and Y are multiples of 3, which means that D is also a multiple of 3.)

ANSWER: (D)

18. Since $\triangle PQR$ is isosceles, then $\angle PRQ = \angle PQR = 2x^\circ$.
Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 2x^\circ$.
Since $\triangle RST$ is isosceles with $RS = RT$, then

$$\angle RST = \tfrac{1}{2}(180^\circ - \angle SRT) = \tfrac{1}{2}(180^\circ - 2x^\circ) = 90^\circ - x^\circ = (90 - x)^\circ$$

ANSWER: (E)

19. We write a general three-digit positive integer in terms of its digits as ABC.
There are 9 possible values for the digit A (the digits 1 to 9) and 10 possible values for each of B and C (the digits 0 to 9).
We want to count the number of such integers with exactly one even digit. We consider the three cases separately.
Suppose that A is even. In this case, B and C are odd.
There are 4 possible values of A (2, 4, 6, 8) and 5 possible values for each of B and C (1, 3, 5, 7, 9). This means that there are $4 \times 5 \times 5 = 100$ integers in this case.
Suppose that B is even. In this case, A and C are odd.
There are 5 possible values of B (0, 2, 4, 6, 8) and 5 possible values for each of A and C (1, 3, 5, 7, 9). This means that there are $5 \times 5 \times 5 = 125$ integers in this case.
Suppose that C is even. In this case, A and B are odd.
There are 5 possible values of C (0, 2, 4, 6, 8) and 5 possible values for each of A and B (1, 3, 5, 7, 9). This means that there are $5 \times 5 \times 5 = 125$ integers in this case. (This case is very similar to the second case.)
Therefore, there are $100 + 125 + 125 = 350$ such integers in total.

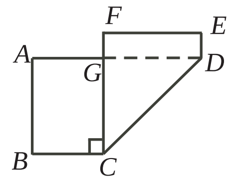
ANSWER: (A)

20. *Solution 1*
Note that $n^{200} = (n^2)^{100}$ and $3^{500} = (3^5)^{100}$.
Since n is a positive integer, then $n^{200} < 3^{500}$ is equivalent to $n^2 < 3^5 = 243$.
Note that $15^2 = 225$, $16^2 = 256$ and if $n \geq 16$, then $n^2 \geq 256$.
Therefore, the largest possible value of n is 15.

Solution 2
Since n is a positive integer and $500 = 200(2.5)$, then $n^{200} < 3^{500}$ is equivalent to $n^{200} < (3^{2.5})^{200}$, which is equivalent to $n < 3^{2.5} = 3^2 3^{0.5} = 9\sqrt{3}$.
Since $9\sqrt{3} \approx 15.59$ and n is an integer, the largest possible value of n is 15.

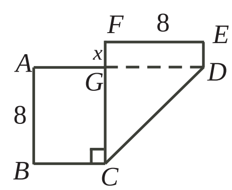
ANSWER: (C)

21. *Solution 1*
The area of the original piece of paper is $17(8) = 136$ cm^2.



When the paper is folded in this way, the portion of the original bottom face of the paper that is visible has the same area as the original portion of the top side of the paper to the right of the fold. (This is quadrilateral $CDEF$.)
Of the portion of the original sheet to the left of the fold, the part that is hidden (and thus not included in the area of the new figure) is the triangular portion under the folded part. (This is the section under $\triangle CDG$.) The hidden triangle is congruent to $\triangle CDG$.
Thus, the area of the portion of the original top face of the paper that is visible is the area to the left of the fold, minus the area of the hidden triangle.
Therefore, the area of the new figure equals the area of the original rectangle minus the area of $\triangle CDG$.
$\triangle CDG$ has height $GC = 8$ cm (the height of the rectangle), is right-angled (since the folded portion of the original bottom edge is perpendicular to the top and bottom edges), and has base $GD = 8$ cm.
Therefore, $\triangle CDG$ has area $\frac{1}{2}(8)(8) = 32$ cm^2.
In summary, the area of the new figure is $136 - 32 = 104$ cm^2.

Solution 2
We show the hidden portion of the original top edge and label the vertices of the new figure:



We want to determine the area of figure $ABCDEFG$.
Suppose that $FG = x$.
Since quadrilateral $AGCB$ has three right angles (at A, B and C), then it is a rectangle, and FC is perpendicular to AD.
Since quadrilateral $DEFG$ has three right angles (at E, F and G), then it is a rectangle.
We know that $FE = AB = 8$ since both are heights of the original rectangle.
Since $AGCB$ and $DEFG$ are rectangles, then $GC = AB = 8$ and $GD = FE = 8$.
Note that BC, CG and GF made up the original bottom edge of the rectangle.
Thus, $BC + CG + GF = 17$, and so $BC = 17 - 8 - x = 9 - x$.
The area of rectangle $AGCB$ is $8(9 - x) = 72 - 8x$.
The area of rectangle $DEFG$ is $8x$.
The area of $\triangle CGD$ (which is right-angled at G) is $\frac{1}{2}(8)(8) = 32$.
Therefore, the total area of figure $ABCDEFG$ is $(72 - 8x) + 8x + 32 = 104$ cm^2.

ANSWER: (A)

22. *Solution 1*
We label the terms $x_1, x_2, x_3, \ldots, x_{2009}, x_{2010}$.
Suppose that S is the sum of the odd-numbered terms in the sequence; that is,

$$S = x_1 + x_3 + x_5 + \cdots + x_{2007} + x_{2009}$$

We know that the sum of all of the terms is 5307; that is,

$$x_1 + x_2 + x_3 + \cdots + x_{2009} + x_{2010} = 5307$$

Next, we pair up the terms: each odd-numbered term with the following even-numbered term. That is, we pair the first term with the second, the third term with the fourth, and so on, until we pair the 2009th term with the 2010th term. There are 1005 such pairs.
In each pair, the even-numbered term is one bigger than the odd-numbered term. That is, $x_2 - x_1 = 1$, $x_4 - x_3 = 1$, and so on.
Therefore, the sum of the even-numbered terms is 1005 greater than the sum of the odd-numbered terms. Thus, the sum of the even-numbered terms is $S + 1005$.
Since the sum of all of the terms equals the sum of the odd-numbered terms plus the sum of the even-numbered terms, then $S + (S + 1005) = 5307$ or $2S = 4302$ or $S = 2151$.
Thus, the required sum is 2151.

Solution 2
Suppose that the first term is x.
Since each term after the first is 1 larger than the previous term, we can label the terms in the sequence as $x, x+1, x+2, \ldots, x+2008, x+2009$.
Since the sum of all 2010 terms is 5307, then

$$\begin{aligned} x + (x+1) + (x+2) + \cdots + (x+2008) + (x+2009) &= 5307 \\ 2010x + (1 + 2 + \cdots + 2008 + 2009) &= 5307 \\ 2010x + \tfrac{1}{2}(2009)(2010) &= 5307 \\ 2010x &= -2013738 \end{aligned}$$

When we add up every second term beginning with the first term, we obtain

$$\begin{aligned} & x + (x+2) + (x+4) + \cdots + (x+2006) + (x+2008) \\ &= 1005x + (2 + 4 + \cdots + 2006 + 2008) \\ &= 1005x + 2(1 + 2 + \cdots + 1003 + 1004) \\ &= 1005x + 2\left(\tfrac{1}{2}(1004)(1005)\right) \\ &= \tfrac{1}{2}(2010x) + (1004)(1005) \\ &= \tfrac{1}{2}(-2013738) + (1004)(1005) \\ &= 2151 \end{aligned}$$

Therefore, the required sum is 2151.

Answer: (C)

23. *Solution 1*
Connie gives 24 bars that account for 45% of the total weight to Brennan. Thus, each of these 24 bars accounts for an average of $\frac{45}{24}\% = \frac{15}{8}\% = 1.875\%$ of the total weight.
Connie gives 13 bars that account for 26% of the total weight to Maya. Thus, each of these 13 bars accounts for an average of $\frac{26}{13}\% = 2\%$ of the total weight.
Since each of the bars that she gives to Blair is heavier than each of the bars given to Brennan (which were the 24 lightest bars) and is lighter than each of the bars given to Maya (which were the 13 heaviest bars), then the average weight of the bars given to Blair must be larger than 1.875% and smaller than 2%.

Note that the bars given to Blair account for $100\% - 45\% - 26\% = 29\%$ of the total weight.
If there were 14 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{14}\% \approx 2.07\%$, which is too large. Thus, there must be more than 14 bars accounting for 29% of the total weight.
If there were 15 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{15}\% \approx 1.93\%$, which is in the correct range.
If there were 16 bars accounting for 29% of the total weight, the average weight would be $\frac{29}{16}\% \approx 1.81\%$, which is too small. The same would be true if there were 17 or 18 bars.
Therefore, Blair must have received 15 bars.

Solution 2
We may assume without loss of generality that the total weight of all of the bars is 100.
Therefore, the bars given to Brennan weigh 45 and the bars given to Maya weigh 26.
Suppose that Connie gives n bars to Blair.
These bars weigh $100 - 45 - 26 = 29$.
Let $b_1, b_2, \ldots, b_{24}$ be the weights of the 24 bars given to Brennan.
We may assume that $b_1 < b_2 < \cdots < b_{23} < b_{24}$, since the weights are all different.
Let $m_1, m_2, \ldots, m_{13}$ be the weights of the 13 bars given to Maya.
We may assume that $m_1 < m_2 < \cdots < m_{12} < m_{13}$, since the weights are all different.
Let $x_1, x_2, \ldots, x_n$ be the weights of the n bars given to Blair.
We may assume that $x_1 < x_2 < \cdots < x_{n-1} < x_n$, since the weights are all different.
Note that

$$b_1 < b_2 < \cdots < b_{23} < b_{24} < x_1 < x_2 < \cdots < x_{n-1} < x_n < m_1 < m_2 < \cdots < m_{12} < m_{13}$$

since the lightest bars are given to Brennan and the heaviest to Maya.
Also,

$$\begin{aligned} b_1 + b_2 + \cdots + b_{23} + b_{24} &= 45 \\ x_1 + x_2 + \cdots + x_{n-1} + x_n &= 29 \\ m_1 + m_2 + \cdots + m_{12} + m_{13} &= 26 \end{aligned}$$

Now b_{24} is the heaviest of the bars given to Brennan, so

$$45 = b_1 + b_2 + \cdots + b_{23} + b_{24} < b_{24} + b_{24} + \cdots + b_{24} + b_{24} = 24b_{24}$$

and so $b_{24} > \frac{45}{24} = \frac{15}{8}$.
Also, m_1 is the lightest of the bars given to Maya, so

$$26 = m_1 + m_2 + \cdots + m_{12} + m_{13} > m_1 + m_1 + \cdots + m_1 + m_1 = 13m_1$$

and so $m_1 < \frac{26}{13} = 2$.
But each of the n bars given to Blair is heavier than b_{24} and each is lighter than m_1.
Thus, $nb_{24} < x_1 + x_2 + \cdots + x_{n-1} + x_n < nm_1$ or $nb_{24} < 29 < nm_1$.
Thus, $\frac{15}{8}n < nb_{24} < 29 < nm_1 < 2n$ and so $n < 29\left(\frac{8}{15}\right) = \frac{232}{15} = 15\frac{7}{15}$ and $n > \frac{29}{2} = 14\frac{1}{2}$.
Since n is an integer, then $n = 15$, so Blair receives 15 bars.

ANSWER: (B)

24. Since the grid is a 5 by 5 grid of squares and each square has side length 10 cm, then the whole grid is 50 cm by 50 cm.

Since the diameter of the coin is 8 cm, then the radius of the coin is 4 cm.
We consider where the centre of the coin lands when the coin is tossed, since the location of the centre determines the position of the coin.
Since the coin lands so that no part of it is off of the grid, then the centre of the coin must land at least 4 cm (1 radius) away from each of the outer edges of the grid.
This means that the centre of the coin lands anywhere in the region extending from 4 cm from the left edge to 4 cm from the right edge (a width of $50 - 4 - 4 = 42$ cm) and from 4 cm from the top edge to 4 cm to the bottom edge (a height of $50 - 4 - 4 = 42$ cm).
Thus, the centre of the coin must land in a square that is 42 cm by 42 cm in order to land so that no part of the coin is off the grid.
Therefore, the total admissible area in which the centre can land is $42 \times 42 = 1764$ cm^2.
Consider one of the 25 squares. For the coin to lie completely inside the square, its centre must land at least 4 cm from each edge of the square.
As above, it must land in a region of width $10 - 4 - 4 = 2$ cm and of height $10 - 4 - 4 = 2$ cm.
There are 25 possible such regions (one for each square) so the area in which the centre of the coin can land to create a winning position is $25 \times 2 \times 2 = 100$ cm^2.
Thus, the probability that the coin lands in a winning position is equal to the area of the region in which the centre lands giving a winning position, divided by the area of the region in which the coin may land, or $\frac{100}{1764} = \frac{25}{441}$.

Answer: (A)

25. First, we define $u(n)$ to be the units digit of the positive integer n (for example, $u(25) = 5$). Next, we make three important notes:

- It is the final position on the circle that we are seeking, not the total number of times travelled around the circle. Therefore, moving 25 steps, for example, around the circle is equivalent to moving 5 steps around the circle because in both cases the counter ends up in the same position. Since 10 steps gives one complete trip around the circle, then we only care about the units digit of the number of steps (that is, $u(n^n)$).
- To determine the final position, we want to determine the sum of the number of steps for each of the 1234 moves; that is, we want to determine

$$S = 1^1 + 2^2 + \cdots + 1233^{1233} + 1234^{1234}$$

since to calculate the position after a move we add the number of steps to the previous position. In fact, as above in the first note, we only care about the units digit of the sum of the numbers of steps $u(S)$, which is equal to

$$u\Big(u(1^1) + u(2^2) + \cdots + u(1233^{1233}) + u(1234^{1234})\Big)$$

- To calculate $u(n^n)$, we only need to worry about the units digit of the base n, $u(n)$, and not the other digits of the base. In other words, we need to determine the units digit of the product of n factors $u(n) \times u(n) \times \cdots \times u(n)$ – namely, $u\Big((u(n))^n\Big)$. (As we will see below, we cannot necessarily truncate the exponent n to its units digit.) To actually perform this calculation, we can always truncate to the units digit at each step because only the units digits affect the units digits.
For example, to calculate $u(13^{13})$, we can calculate $u\Big((u(13))^{13}\Big)$ which equals $u(3^{13})$. In

other words, we need to calculate $3 \times 3 \times \cdots \times 3$, which we can do by multiplying and truncating at each step if necessary:

$$3, 9, 27 \to 7, 21 \to 1, 3, 9, 27 \to 7, 21 \to 1, 3, 9, 27 \to 7, 21 \to 1, 3$$

Thus, $u(13^{13}) = 3$.

Next, we consider the different possible values of $u(n)$ and determine a pattern of the units digits of powers of n:

- If $u(n)$ is 0, 1, 5, or 6, then $u(n^k)$ is 0, 1, 5, or 6, respectively, for any positive integer k, because $u(n^2) = u(n)$ in each case, and so raising to higher powers does not affect the units digit.
- If $u(n) = 4$, then the units digits of powers of n alternate $4, 6, 4, 6$, and so on. (We can see by multiplying and truncating as above.)
- If $u(n) = 9$, then the units digits of powers of n alternate $9, 1, 9, 1$, and so on.
- If $u(n) = 2$, then the units digits of powers of n cycle as $2, 4, 8, 6, 2, 4, 8, 6$ and so on.
- If $u(n) = 8$, then the units digits of powers of n cycle as $8, 4, 2, 6, 8, 4, 2, 6$ and so on.
- If $u(n) = 3$, then the units digits of powers of n cycle as $3, 9, 7, 1, 3, 9, 7, 1$ and so on.
- If $u(n) = 7$, then the units digits of powers of n cycle as $7, 9, 3, 1, 7, 9, 3, 1$ and so on.

Next, we determine $u(n^n)$, based on $u(n)$:

- If $u(n)$ is 0, 1, 5, or 6, then $u(n^n)$ is 0, 1, 5, or 6, respectively.
- If $u(n) = 4$, then $u(n^n) = 6$, since the exponent is even so the units digit will be that occurring in even positions in the pattern.
- If $u(n) = 9$, then $u(n^n) = 9$, since the exponent is odd so the units digit will be that occurring in odd positions in the pattern.
- If $u(n) = 2$, then $u(n^n)$ will be either 4 or 6, depending on the exponent n, because the exponent is certainly even, but the pattern of units digits cycles with length 4. In particular, $u(2^2) = 4$ and $u(12^{12}) = 6$ (the exponent 12 is a multiple of 4, so the units digit appears at the end of a cycle).
- If $u(n) = 8$, then $u(n^n)$ will be either 4 or 6, depending on the exponent n, because the exponent is certainly even, but the pattern of units digits cycles with length 4. In particular, $u(8^8) = 6$ and $u(18^{18}) = 4$, using a similar argument to above.
- Similarly, $u(3^3) = 7$, $u(13^{13}) = 3$, $u(7^7) = 3$, and $u(17^{17}) = 7$.
- Since the units digits of the base n repeat in a cycle of length 10 and the units digits of the powers of n for a fixed n repeat every 1, 2 or 4 powers, then $u(n^n)$ repeats in a cycle of length 20 (the least common multiple of 10, 1, 2, and 4).

We can now determine $u(1) + u(2) + \cdots + u(19) + u(20)$ to be

$$u(1+4+7+6+5+6+3+6+9+0+1+6+3+6+5+6+7+4+9+0) = u(94) = 4$$

To calculate the total for 1234 steps, we note that 61 cycles of 20 bring us to 1220 steps. After 1220 steps, the units digit of the sum is $u(61 \cdot 4) = u(244) = 4$.

We then add the units digit of the sum of 14 more steps, starting at the beginning of the cycle, to obtain a final position equal to the units digit of

$$u(4 + (1 + 4 + 7 + 6 + 5 + 6 + 3 + 6 + 9 + 0 + 1 + 6 + 3 + 6)) = u(67) = 7$$

Therefore, the final position is 7.

Answer: (D)