Canadian
Mathematics
Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Pascal Contest

(Grade 9)

Wednesday, February 18, 2009

Solutions

1. Calculating, $2 \times 9 - \sqrt{36} + 1 = 18 - 6 + 1 = 13$.

ANSWER: (D)

2. On Saturday, Deepit worked 6 hours. On Sunday, he worked 4 hours.
Therefore, he worked $6 + 4 = 10$ hours in total on Saturday and Sunday.

ANSWER: (E)

3. Since 1 piece of gum costs 1 cent, then 1000 pieces of gum cost 1000 cents.
Since there are 100 cents in a dollar, the total cost is \$10.00.

ANSWER: (D)

4. Since each of the 18 classes has 28 students, then there are $18 \times 28 = 504$ students who attend the school.
On Monday, there were 496 students present, so $504 - 496 = 8$ students were absent.

ANSWER: (A)

5. The sum of the angles around any point is 360°.
Therefore, $5x^\circ + 4x^\circ + x^\circ + 2x^\circ = 360^\circ$ or $12x = 360$ or $x = 30$.

ANSWER: (D)

6. When -1 is raised to an even exponent, the result is 1.
When -1 is raised to an odd exponent, the result is -1.
Thus, $(-1)^5 - (-1)^4 = -1 - 1 = -2$.

ANSWER: (A)

7. Since PQ is horizontal and the y-coordinate of P is 1, then the y-coordinate of Q is 1.
Since QR is vertical and the x-coordinate of R is 5, then the x-coordinate of Q is 5.
Therefore, the coordinates of Q are $(5, 1)$.

ANSWER: (C)

8. When $y = 3$, we have $\dfrac{y^3 + y}{y^2 - y} = \dfrac{3^3 + 3}{3^2 - 3} = \dfrac{27 + 3}{9 - 3} = \dfrac{30}{6} = 5$.

ANSWER: (D)

9. Since there are 4 ♣'s in each of the first two columns, then at least 1 ♣ must be moved out of each of these columns to make sure that each column contains exactly three ♣'s.
Therefore, we need to move at least 2 ♣'s in total.
If we move the ♣ from the top left corner to the bottom right corner

	♣	♣	♣	
♣	♣	♣		♣
♣	♣			
♣	♣		♣	
		♣	♣	♣

and the ♣ from the second row, second column to the third row, fifth column

	♣	♣	♣	
♣		♣		♣
♣	♣			♣
♣	♣		♣	
		♣	♣	♣

then we have exactly three ♣'s in each row and each column.

Therefore, since we must move at least 2 ♣'s and we can achieve the configuration that we want by moving 2 ♣'s, then 2 is the smallest number.
(There are also other combinations of moves that will give the required result.)

Answer: (B)

10. *Solution 1*
Since $z = 4$ and $x + y = 7$, then $x + y + z = (x + y) + z = 7 + 4 = 11$.

Solution 2
Since $z = 4$ and $x + z = 8$, then $x + 4 = 8$ or $x = 4$.
Since $x = 4$ and $x + y = 7$, then $4 + y = 7$ or $y = 3$.
Therefore, $x + y + z = 4 + 3 + 4 = 11$.

Answer: (C)

11. We write out the five numbers to 5 decimal places each, without doing any rounding:

$$\begin{aligned} 5.07\overline{6} &= 5.07666\ldots \\ 5.0\overline{76} &= 5.07676\ldots \\ 5.07 &= 5.07000 \\ 5.076 &= 5.07600 \\ 5.\overline{076} &= 5.07607\ldots \end{aligned}$$

We can use these representations to order the numbers as

$$5.07000, 5.07600, 5.07607\ldots, 5.07666\ldots, 5.07676\ldots$$

so the number in the middle is $5.\overline{076}$.

Answer: (E)

12. *Solution 1*
Since there are 24 hours in a day, Francis spends $\frac{1}{3} \times 24 = 8$ hours sleeping.
Also, he spends $\frac{1}{4} \times 24 = 6$ hours studying, and $\frac{1}{8} \times 24 = 3$ hours eating.
The number of hours that he has left is $24 - 8 - 6 - 3 = 7$ hours.

Solution 2
Francis spends $\frac{1}{3} + \frac{1}{4} + \frac{1}{8} = \frac{8+6+3}{24} = \frac{17}{24}$ of a day either sleeping, studying or eating.
This leaves him $1 - \frac{17}{24} = \frac{7}{24}$ of his day.
Since there are 24 hours in a full day, then he has 7 hours left.

Answer: (D)

13. *Solution 1*
Since the sum of the angles in a triangle is 180°, then

$$\angle QPS = 180^\circ - \angle PQS - \angle PSQ = 180^\circ - 48^\circ - 38^\circ = 94^\circ$$

Therefore, $\angle RPS = \angle QPS - \angle QPR = 94^\circ - 67^\circ = 27^\circ$.

Solution 2
Since the sum of the angles in a triangle is 180°, then

$$\angle QRP = 180^\circ - \angle PQR - \angle QPR = 180^\circ - 48^\circ - 67^\circ = 65^\circ$$

Therefore, $\angle PRS = 180^\circ - \angle PRQ = 180^\circ - 65^\circ = 115^\circ$.
Using $\triangle PRS$,

$$\angle RPS = 180^\circ - \angle PRS - \angle PSR = 180^\circ - 115^\circ - 38^\circ = 27^\circ$$

Answer: (A)

14. The perimeter of the shaded region equals the sum of the lengths of OP and OQ plus the length of arc PQ.
Each of OP and OQ has length 5.
Arc PQ forms $\frac{3}{4}$ of the circle with centre O and radius 5, because the missing portion corresponds to a central angle of 90°, and so is $\frac{1}{4}$ of the total circle.
Thus, the length of arc PQ is $\frac{3}{4}$ of the circumference of this circle, or $\frac{3}{4}(2\pi(5)) = \frac{15}{2}\pi$.
Therefore, the perimeter is $5 + 5 + \frac{15}{2}\pi \approx 33.56$ which, of the given answers, is closest to 34.
Answer: (A)

15. After some trial and error, we obtain the two lists $\{4, 5, 7, 8, 9\}$ and $\{3, 6, 7, 8, 9\}$.
Why are these the only two?

If the largest number of the five integers was 8, then the largest that the sum could be would be $8 + 7 + 6 + 5 + 4 = 30$, which is too small. This tells us that we must include one 9 in the list. (We cannot include any number larger than 9, since each number must be a single-digit number.)
Therefore, the sum of the remaining four numbers is $33 - 9 = 24$.

If the largest of the four remaining numbers is 7, then their largest possible sum would be $7 + 6 + 5 + 4 = 22$, which is too small. Therefore, we also need to include an 8 in the list.
Thus, the sum of the remaining three numbers is $24 - 8 = 16$.

If the largest of the three remaining numbers is 6, then their largest possible sum would be $6 + 5 + 4 = 15$, which is too small. Therefore, we also need to include an 7 in the list.
Thus, the sum of the remaining two numbers is $16 - 7 = 9$.

This tells us that we need two different positive integers, each less than 7, that add to 9. These must be 3 and 6 or 4 and 5.
This gives us the two lists above, and shows that they are the only two such lists.
Answer: (B)

16. The area of the entire grid is $4 \times 9 = 36$.
The area of $\triangle PQR$ is $\frac{1}{2}(QR)(PQ) = \frac{1}{2}(3)(4) = 6$.
The area of $\triangle STU$ is $\frac{1}{2}(ST)(UT) = \frac{1}{2}(4)(3) = 6$.
The area of the rectangle with base RS is $2 \times 4 = 8$.
Therefore, the total shaded area is $6 + 6 + 8 = 20$ and so the unshaded area is $36 - 20 = 16$.
The ratio of the shaded area to the unshaded area is $20 : 16 = 5 : 4$.
Answer: (E)

17. We can suppose that each test is worth 100 marks.
Since the average of her five test marks is 73%, then the total number of marks that she received is $5 \times 73 = 365$.
Once her teacher removes a mark, her new average is 76% so the sum of the remaining four marks is $4 \times 76 = 304$.
Since $365 - 304 = 61$, then the mark removed was 61%.
Answer: (B)

18. *Solution 1*
From December 31, 1988 to December 31, 2008, a total of 20 years have elapsed.
A time period of 20 years is the same as five 4 year periods.

Thus, the population of Arloe has doubled 5 times over this period to its total of 3456.
Doubling 5 times is equivalent to multiplying by $2^5 = 32$.
Therefore, the population of Arloe on December 31, 1988 was $\dfrac{3456}{32} = 108$.

Solution 2
The population doubles every 4 years going forward, so is halved every 4 years going backwards in time.
The population on December 31, 2008 was 3456.
The population on December 31, 2004 was $3456 \div 2 = 1728$.
The population on December 31, 2000 was $1728 \div 2 = 864$.
The population on December 31, 1996 was $864 \div 2 = 432$.
The population on December 31, 1992 was $432 \div 2 = 216$.
The population on December 31, 1988 was $216 \div 2 = 108$.

Answer: (D)

19. Since Pat drives 60 km at 80 km/h, this takes him $\dfrac{60 \text{ km}}{80 \text{ km/h}} = \frac{3}{4}$ h.
Since Pat has 2 hours in total to complete the trip, then he has $2 - \frac{3}{4} = \frac{5}{4}$ hours left to complete the remaining $150 - 60 = 90$ km.
Therefore, he must travel at $\dfrac{90 \text{ km}}{\frac{5}{4} \text{ h}} = \frac{360}{5}$ km/h $= 72$ km/h.

Answer: (C)

20. Since the three numbers in each straight line must have a product of 3240 and must include 45, then the other two numbers in each line must have a product of $\frac{3240}{45} = 72$.
The possible pairs of positive integers are 1 and 72, 2 and 36, 3 and 24, 4 and 18, 6 and 12, and 8 and 9.
The sums of the numbers in these pairs are 73, 38, 27, 22, 18, and 17.
To maximize the sum of the eight numbers, we want to choose the pairs with the largest possible sums, so we choose the first four pairs.
Thus, the largest possible sum of the eight numbers is $73 + 38 + 27 + 22 = 160$.

Answer: (E)

21. Since each of Alice and Bob rolls one 6-sided die, then there are $6 \times 6 = 36$ possible combinations of rolls.
Each of these 36 possibilities is equally likely.
Alice wins when the two values rolled differ by 1. The possible combinations that differ by 1 are $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, $(2, 1)$, $(3, 2)$, $(4, 3)$, $(5, 4)$, and $(6, 5)$.
Therefore, there are 10 combinations when Alice wins.
Thus her probability of winning is $\frac{10}{36} = \frac{5}{18}$.

Answer: (C)

22. Diameters PQ and RS cross at the centre of the circle, which we call O.
The area of the shaded region is the sum of the areas of $\triangle POS$ and $\triangle ROQ$ plus the sum of the areas of sectors POR and SOQ.
Each of $\triangle POS$ and $\triangle ROQ$ is right-angled and has its two perpendicular sides of length 4 (the radius of the circle).
Therefore, the area of each of these triangles is $\frac{1}{2}(4)(4) = 8$.
Each of sector POR and sector SOQ has area $\frac{1}{4}$ of the total area of the circle, as each has

central angle 90° (that is, $\angle POR = \angle SOQ = 90^\circ$) and 90° is one-quarter of the total central angle.
Therefore, each sector has area $\frac{1}{4}(\pi(4^2)) = \frac{1}{4}(16\pi) = 4\pi$.
Thus, the total shaded area is $2(8) + 2(4\pi) = 16 + 8\pi$.

ANSWER: (E)

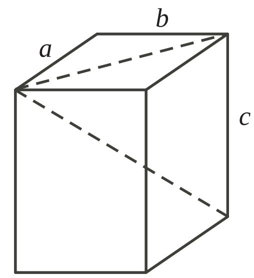
23. The maximum possible mass of a given coin is $7 \times (1 + 0.0214) = 7 \times 1.0214 = 7.1498$ g.
The minimum possible mass of a given coin is $7 \times (1 - 0.0214) = 7 \times 0.9786 = 6.8502$ g.
What are the possible numbers of coins that could make up 1000 g?
To find the largest number of coins, we want the coins to be as light as possible. If all of the coins were as light as possible, we would have $\dfrac{1000}{6.8502} \approx 145.98$ coins. Now, we cannot have a non-integer number of coins. This means that we must have at most 145 coins. (If we had 146 coins, the total mass would have to be at least $146 \times 6.8502 = 1000.1292$ g, which is too heavy.) Practically, we can get 145 coins to have a total mass of 1000 g by taking 145 coins at the minimum possible mass and making each slightly heavier.
To find the smallest number of coins, we want the coins to be as heavy as possible. If all of the coins were as heavy as possible, we would have $\dfrac{1000}{7.1498} \approx 139.86$ coins. Again, we cannot have a non-integer number of coins. This means that we must have at least 140 coins. (If we had 139 coins, the total mass would be at most $139 \times 7.1498 = 993.8222$ g, which is too light.)
Therefore, the difference between the largest possible number and smallest possible number of coins is $145 - 140 = 5$.

ANSWER: (B)

24. Divide the large cube of side length 40 into 8 smaller cubes of side length 20, by making three cuts of the large cube through its centre using planes parallel to the pairs of faces.
Each of these small cubes has the centre of the large cube as its vertex.
Each of these small cubes also just encloses one of the large spheres, in the sense that the sphere just touches each of the faces of the small cube.
We call the sphere that fits in the central space the inner sphere. To make this sphere as large possible, its centre will be at the centre of the large cube. (If this was not the case, the centre be outside one of the small cubes, and so would be farther away from one of the large spheres than from another.)
To find the radius of the inner sphere, we must find the shortest distance from the centre of the large cube (that is, the centre of the inner sphere) to one of the large spheres. (Think of starting the inner sphere as a point at the centre of the cube and inflating it until it just touches the large spheres.)

Consider one of these small cubes and the sphere inside it.
Join the centre of the small cube to one of its vertices.
Since the small cube has side length 20, then this new segment has length $\sqrt{10^2 + 10^2 + 10^2}$ or $\sqrt{300}$, since to get from the centre to a vertex, we must go over 10, down 10 and across 10. (See below for an explanation of why this distance is thus $\sqrt{300}$.)
The inner sphere will touch the large sphere along this segment.
Thus, the radius of the inner sphere will be this distance ($\sqrt{300}$) minus the radius of the large sphere (10), and so is $\sqrt{300} - 10 \approx 7.32$.
Of the given answers, this is closest to 7.3.

(We need to justify why the distance from the centre of the small cube to its vertex is $\sqrt{10^2 + 10^2 + 10^2}$.
Divide the small cube into 8 tiny cubes of side length 10 each. The distance from the centre of the small cube to its vertex is equal to the length of a diagonal of one of the tiny cubes.
Consider a rectangular prism with edge lengths a, b and c. What is the length, d, of the diagonal inside the prism?
By the Pythagorean Theorem, a face with side lengths a and b has a diagonal of length $\sqrt{a^2 + b^2}$.
Consider the triangle formed by this diagonal, the diagonal of the prism and one of the vertical edges of the prism, of length c.



This triangle is right-angled since the vertical edge is perpendicular to the top face. By the Pythagorean Theorem again, $d^2 = (\sqrt{a^2 + b^2})^2 + c^2$, so $d^2 = a^2 + b^2 + c^2$ or $d = \sqrt{a^2 + b^2 + c^2}$.)

ANSWER: (B)

25. The three machines operate in a way such that if the two numbers in the output have a common factor larger than 1, then the two numbers in the input would have to have a common factor larger than 1.
To see this, let us look at each machine separately. We use the fact that if two numbers are each multiples of d, then their sum and difference are also multiples of d.
Suppose that (m, n) is input into Machine A. The output is (n, m). If n and m have a common factor larger than 1, then m and n do as well.
Suppose that (m, n) is input into Machine B. The output is $(m + 3n, n)$. If $m + 3n$ and n have a common factor d, then $(m + 3n) - n - n - n = m$ has a factor of d as each part of the subtraction is a multiple of d. Therefore, m and n have a common factor of d.
Suppose that (m, n) is input into Machine C. The output is $(m - 2n, n)$. If $m - 2n$ and n have a common factor d, then $(m - 2n) + n + n = m$ has a factor of d as each part of the addition is a multiple of d. Therefore, m and n have a common factor of d.
In each case, any common factor that exists in the output is present in the input.

Let us look at the numbers in the five candidates.
After some work, we can find the prime factorizations of the six integers:

$$\begin{aligned}
2009 &= 7(287) = 7(7)(41)\\
1016 &= 8(127) = 2(2)(2)(127)\\
1004 &= 4(251) = 2(2)(251)\\
1002 &= 2(501) = 2(3)(167)\\
1008 &= 8(126) = 8(3)(42) = 16(3)(3)(7) = 2(2)(2)(2)(3)(3)(7)\\
1032 &= 8(129) = 8(3)(43) = 2(2)(2)(3)(43)
\end{aligned}$$

Therefore, the only one of 1002, 1004, 1008, 1016, 1032 that has a common factor larger than 1 with 2009 is 1008, which has a common factor of 7 with 2009.

How does this help? Since 2009 and 1008 have a common factor of 7, then whatever pair was input to produce $(2009, 1008)$ must have also had a common factor of 7. Also, the pair that was input to create this pair also had a common factor of 7. This can be traced back through every step to say that the initial pair that produces the eventual output of $(2009, 1008)$ must have a common factor of 7.
Thus, $(2009, 1008)$ cannot have come from $(0, 1)$.

Notes:

- This does not tell us that the other four pairs necessarily work. It does tell us, though, that $(2009, 1008)$ cannot work.
- We can trace the other four outputs back to $(0, 1)$ with some effort. (This process is easier to do than it is to describe!)
  To do this, we notice that if the output of Machine A was (a, b), then its input was (b, a), since Machine A switches the two entries.
  Also, if the output of Machine B was (a, b), then its input was $(a - 3b, b)$, since Machine B adds three times the second number to the first.
  Lastly, if the output of Machine C was (a, b), then its input was $(a + 2b, b)$, since Machine C subtracts two times the second number from the first.
  Consider $(2009, 1016)$ for example. We try to find a way from $(2009, 1016)$ back to $(0, 1)$. We only need to find one way that works, rather than looking for a specific way.

  We note before doing this that starting with an input of (m, n) and then applying Machine B then Machine C gives an output of $((m + 3n) - 2n, n) = (m + n, n)$. Thus, if applying Machine B then Machine C (we call this combination "Machine BC") gives an output of (a, b), then its input must have been $(a - b, b)$. We can use this combined machine to try to work backwards and arrive at $(0, 1)$. This will simplify the process and help us avoid negative numbers.
  We do this by making a chart and by attempting to make the larger number smaller wherever possible:

Output	Machine	Input
$(2009, 1016)$	BC	$(993, 1016)$
$(993, 1016)$	A	$(1016, 993)$
$(1016, 993)$	BC	$(23, 993)$
$(23, 993)$	A	$(993, 23)$
$(993, 23)$	BC, 43 times	$(4, 23)$
$(4, 23)$	A	$(23, 4)$
$(23, 4)$	BC, 5 times	$(3, 4)$
$(3, 4)$	A	$(4, 3)$
$(4, 3)$	BC	$(1, 3)$
$(1, 3)$	A	$(3, 1)$
$(3, 1)$	B	$(0, 1)$

  Therefore, by going up through this table, we can see a way to get from an initial input of $(0, 1)$ to a final output of $(2009, 1016)$.
  In a similar way, we can show that we can obtain final outputs of each of $(2009, 1004)$, $(2009, 1002)$, and $(2009, 1032)$.

ANSWER: (D)