Canadian
Mathematics
Competition

*An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario*

2007 Hypatia Contest

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Solutions

1. (a) The possible routes are:

$A \to B \to C \to D \to A$ $\quad$ $A \to B \to D \to C \to A$
$A \to C \to B \to D \to A$ $\quad$ $A \to C \to D \to B \to A$
$A \to D \to B \to C \to A$ $\quad$ $A \to D \to C \to B \to A$

(b) We list each route and its length:

$A \to B \to C \to D \to A$: Length $AB + BC + CD + DA = 80 + 120 + 90 + 40 = 330$ km
$A \to B \to D \to C \to A$: Length $AB + BD + DC + CA = 80 + 60 + 90 + 105 = 335$ km
$A \to C \to B \to D \to A$: Length $AC + CB + BD + DA = 105 + 120 + 60 + 40 = 325$ km
$A \to C \to D \to B \to A$: Length $AC + CD + DB + BA = 105 + 90 + 60 + 80 = 335$ km
$A \to D \to B \to C \to A$: Length $AD + DB + BC + CA = 40 + 60 + 120 + 105 = 325$ km
$A \to D \to C \to B \to A$: Length $AD + DC + CB + BA = 40 + 90 + 120 + 80 = 330$ km

The two routes of shortest length are $A \to C \to B \to D \to A$ and $A \to D \to B \to C \to A$, which are each of length 325 km.
The two routes of longest length are $A \to B \to D \to C \to A$ and $A \to C \to D \to B \to A$, which are each of length 335 km.

(c) *Solution 1*
We can list the possible routes:
$A \to B \to C \to E \to D \to A$ $\quad$ $A \to B \to D \to E \to C \to A$
$A \to C \to B \to E \to D \to A$ $\quad$ $A \to C \to D \to E \to B \to A$
$A \to D \to B \to E \to C \to A$ $\quad$ $A \to D \to C \to E \to B \to A$
Therefore, there are 6 possible routes.
(Note that in fact each route from (a) gives a route here in (c) by adding an E between the third and fourth stops on the original route.)

Solution 2
Consider a route $A \to x \to y \to E \to z \to A$.
There are 3 possibilities for x (B, C or D).
For each of these possibilities, there are 2 possibilities for y.
After x and y are chosen, there is only 1 possibility of z.
So there are $3 \times 2 = 6$ possible routes.

(d) From the first piece of information, $AD + DC + CE + EB + BA = 600$ km so $40 + 90 + CE + EB + 80 = 600$ km or $CE + EB = 390$ km.
From the second piece of information, $AC + CD + DE + EB + BA = 700$ km so $105 + 90 + 225 + EB + 80 = 700$ km or $EB = 200$ km.
Since $EB = 200$ km and $CE + EB = 390$ km, then $CE = 190$ km, so the distance from C to E is 190 km.

2. (a) Here is a sequence of moves that works:

Move #	P	Q	R	S	Comment
	9	9	1	5	
1	8	8	4	4	3 added to R
2	7	7	7	3	3 added to R
3	6	6	6	6	3 added to S

There are other sequences of moves that will work.

(b) i. In total, there are $31 + 27 + 27 + 7 = 92$ marbles, so if there is an equal number in each pail, there must be 23 in each pail.

Here is a sequence of moves that works:

Move #	P	Q	R	S	Comment
	31	27	27	7	
1	30	26	26	10	3 added to S
2	29	25	25	13	3 added to S
3	28	24	24	16	3 added to S
4	27	23	23	19	3 added to S
5	26	22	22	22	3 added to S
6	25	21	21	25	3 added to S
7	24	24	20	24	3 added to Q
8	23	23	23	23	3 added to R

There are other sequences of moves that will work.

ii. Initially, pail P contains 31 marbles.
We want pail P to contain 23 marbles, so we must decrease the number of marbles in pail P by 8.
In any legal move, the number of marbles in pail P decreases by at most 1 (that is, it decreases by 1 or increases by 3).
Therefore, we need at least 8 legal moves in which the number of marbles in pail P decreases (and potentially some where the number of marbles in pail P increases).
Thus, it takes at least 8 legal moves to obtain the same number of marbles in each pail.
(Note that in part (i), we showed that we could do this in 8 legal moves, so 8 is the minimum number of moves needed.)

(c) *Solution 1*
Starting with 10, 8, 11, and 7 marbles in the pails, there are $10 + 8 + 11 + 7 = 36$ marbles in total.
To have an equal number of marbles in each pail, we would need $36 \div 4 = 9$ marbles in each pail.
On any legal move, the number of marbles in any pail decreases by 1 or increases by 3.
If the pail contains an even number n of marbles before a legal move, then it will contain either $n-1$ or $n+3$ marbles after the legal move, so will contain an odd number of marbles.
Similarly, if the pail contains an odd number of marbles before a legal move, then it will contain an even number of marbles after the legal move.
But we start with two pails containing an even number of marbles and two pails containing an odd number of marbles.
After the first legal move, the pails originally containing an even number of marbles will contain an odd number of marbles and the pails originally containing an odd number of marbles will contain an even number of marbles.
This gets us back to the the same situation – two pails with an even number and two pails with an odd number of marbles.
Therefore, after any move, this situation will not change.
Therefore, it is impossible to ever have 9 marbles in each pail, as there will always be two pails containing an even number of marbles.

Solution 2
Starting with 10, 8, 11, and 7 marbles in the pails, there are $10 + 8 + 11 + 7 = 36$ marbles in total.
To have an equal number of marbles in each pail, we would need $36 \div 4 = 9$ marbles in

each pail.
On any legal move, the number of marbles in any pail decreases by 1 or increases by 3.

Assume that it is possible to end up with 9 marbles in each pail.
We show that this cannot happen by proving the following fact:

> If it is possible to end up with 9 marbles in each pail, then after each move, the difference between the number of marbles in any two pails must be a multiple of 4.

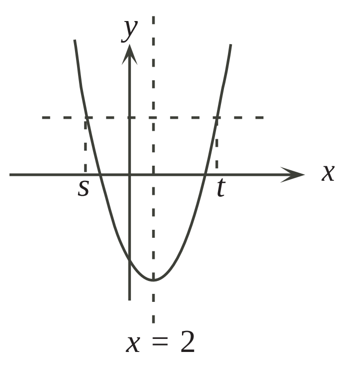
Assume that this fact was true after a certain move. (We know that it is true at the end, since the difference between the numbers in any pair of pails is 0.)
Suppose that there were a, b, c and d marbles in the pails.
Pick two of the four pails (say, the pails with a and b marbles). Before this move (that is, after the previous move), either these two pails each had 1 more marble in each (so $a+1$ and $b+1$ marbles which preserves the difference) or one pail had 1 more marble and the other had 3 fewer marbles (so $a+1$ and $b-3$ or $a-3$ and $b+1$ which changes the difference by 4).
Therefore, before this move, the differences between the number of marbles in the pails are all multiples of 4.
This tells us that, to end up with 9 marbles in each pail, the difference between the numbers of marbles in any pair of pails is always a multiple of 4.
But this is not true with our initial condition of 10, 8, 11 and 7 marbles (since, for example, $11-10$ is not a multiple of 4).
Therefore, it is impossible to end up with an equal number of marbles in each pail.

3. (a) If $f(x)=0$, then $x^2-4x-21=0$.
Factoring the left side, we obtain $(x-7)(x+3)=0$, so $x=7$ or $x=-3$.
(We could obtain the same values of x by using the quadratic formula.)

(b) *Solution 1*
Completing the square in the original function,

$$f(x)=x^2-4x-21=x^2-4x+4-4-21=(x-2)^2-25$$

so the axis of symmetry of the parabola $y=f(x)$ is the vertical line $x=2$. (The axis of symmetry could also have been found using the average of the roots from (a).)
If $f(s)=f(t)$, then s and t are symmetrically located around the axis of symmetry.



In other words, the average value of s and t is the x-coordinate of the axis of symmetry, so $\frac{1}{2}(s+t)=2$ or $s+t=4$.

(Note that this agrees with our answer from part (a), but that we needed to proceed formally here to make sure that there were no other answers.)

Solution 2
Rearranging,

$$\begin{aligned} s^2 - 4s - 21 &= t^2 - 4t - 21 \\ s^2 - t^2 - 4s + 4t &= 0 \\ (s+t)(s-t) - 4(s-t) &= 0 \\ (s+t-4)(s-t) &= 0 \end{aligned}$$

Therefore, $s + t - 4 = 0$ or $s - t = 0$.
Since we are told that s and t are different real numbers, then $s - t \neq 0$.
Therefore, $s + t - 4 = 0$ or $s + t = 4$.

(c) *Solution 1*
Proceeding algebraically in a similar way to part (b), Solution 2,

$$\begin{aligned} (a^2 - 4a - 21) - (b^2 - 4b - 21) &= 4 \\ a^2 - b^2 - 4a + 4b &= 4 \\ (a+b-4)(a-b) &= 4 \end{aligned}$$

Since a and b are integers, then $a+b-4$ and $a-b$ are integers as well. In particular, they are integers whose product is 4.
We make a table to check the possibilities:

$a+b-4$	$a-b$	$2a-4$	a	b
4	1	5	$\frac{9}{2}$	$\frac{7}{2}$
2	2	4	4	2
1	4	5	$\frac{9}{2}$	$\frac{1}{2}$
-4	-1	-5	$-\frac{1}{2}$	$\frac{1}{2}$
-2	-2	-4	0	2
-1	-4	-5	$-\frac{1}{2}$	$\frac{7}{2}$

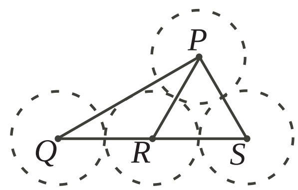
Therefore, the one pairs of positive integer values of a and b that works is $(a, b) = (4, 2)$.
(Note that we could have cut down our work in this table by noticing that if $a+b-4 = x$ and $a - b = y$, then $2a = x + y$, so $x + y$ (that is, the sum of the values of $a + b - 4$ and $a - b$) must be even, which eliminates all but two of the rows in the table.)

Solution 2
As in part (b), the axis of symmetry of the parabola $y = f(x)$ is $x = 2$.
Since the parabola has leading coefficient $+1$, then it is the same shape as the parabola $y = x^2$.
In the parabola $y = x^2$ (and so in the parabola $y = f(x)$), the lattice points moving to the right from the axis of symmetry are $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$, $(4, 16)$, and so on. The vertical distances moving from one point to the next are 1, 3, 5, 7, and so on.
A similar pattern is true when we move successive units to the left from the axis of symmetry.
Starting from the left, the sequence of successive vertical differences is thus

$$\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots$$

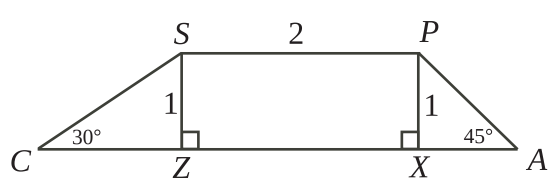
For $f(a) - f(b) = 4$ with a and b integers, we must find a sequence of consecutive differences that add to 4 or -4 (depending on whether a or b is further to the left).
We can only get 4 or -4 by using $(-3)+(-1)$ or $1+3$. The relative positions of these are starting at the axis of symmetry and moving two units to the right, or starting two units to the left of the axis of symmetry and moving two units to the right.
Since the axis of symmetry for the given parabola is $x = 2$, then the only solution is $(a, b) = (4, 2)$, since a and b must both be positive.

4. (a) Join PQ, PR, PS, RQ, and RS.
Since the circles with centre Q, R and S are all tangent to BC, then QR and RS are each parallel to BC (as the centres Q, R and S are each 1 unit above BC).
This tells us that QS passes through R.
When the centres of tangent circles are joined, the line segments formed pass through the associated point of tangency, and so have lengths equal to the sum of the radii of those circles.
Therefore, $QR = RS = PR = PS = 1 + 1 = 2$.



Since $PR = PS = RS$, then $\triangle PRS$ is equilateral, so $\angle PSR = \angle PRS = 60^\circ$.
Since $\angle PRS = 60^\circ$ and QRS is a straight line, then $\angle QRP = 180^\circ - 60^\circ = 120^\circ$.
Since $QR = RP$, then $\triangle QRP$ is isosceles, so $\angle PQR = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$.
Since $\angle PQS = 30^\circ$ and $\angle PSQ = 60^\circ$, then $\angle QPS = 180^\circ - 30^\circ - 60^\circ = 90^\circ$, so $\triangle PQS$ is a 30°-60°-90° triangle.

(b) In (a), we saw that QS is parallel to BC.
Similarly, since P and S are each one unit from AC, then PS is parallel to AC.
Also, since P and Q are each one unit from AB, then PQ is parallel to AB.
Therefore, the sides of $\triangle PQS$ are parallel to the corresponding sides of $\triangle ABC$.
Thus, the angles of $\triangle ABC$ are equal to the corresponding angles of $\triangle PQS$, so $\triangle ABC$ is a 30°-60°-90° triangle.
This means that if we can determine one of the side lengths of $\triangle ABC$, we can then determine the lengths of the other two sides using the side ratios in a 30°-60°-90° triangle.
Consider side AC.
Since the circle with centre P is tangent to sides AB and AC, then the line through A and P bisects $\angle BAC$. Thus, $\angle PAC = 45^\circ$.
Similarly, the line through C and S bisects $\angle ACB$. Thus, $\angle SCA = 30^\circ$.
We extract trapezoid $APSC$ from the diagram, obtaining



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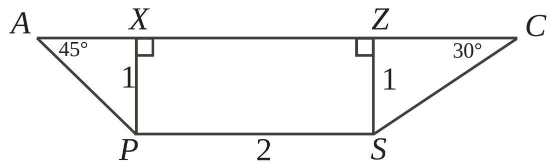
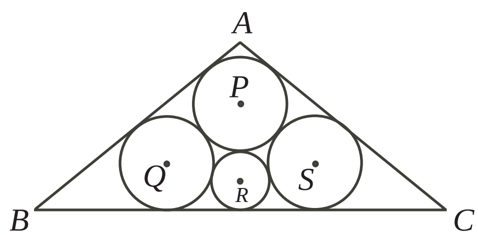
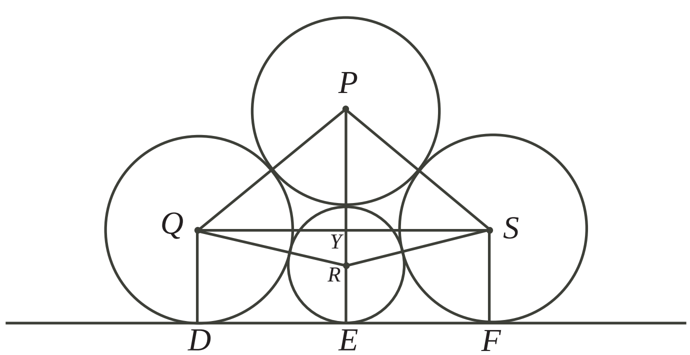


depending on your perspective. Drop perpendiculars from P and S to X and Z on side AC.
Since PS is parallel to AC and PX and SZ are perpendicular to AC, then $PXZS$ is a rectangle, so $XZ = PS = 2$.
Since $\triangle AXP$ is right-angled at X, has $PX = 1$ (the radius of the circle), and $\angle PAX = 45°$, then $AX = PX = 1$.
Since $\triangle CZS$ is right-angled at Z, has $SZ = 1$ (the radius of the circle), and $\angle SCZ = 30°$, then $CZ = \sqrt{3}SZ = \sqrt{3}$ (since $\triangle SZC$ is also a 30°-60°-90° triangle).
Thus, $AC = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$.
Since $\triangle ABC$ is a 30°-60°-90° triangle, with $\angle ACB = 60°$ and $\angle CAB = 90°$, then $BC = 2AC = 6 + 2\sqrt{3}$, and $AB = \sqrt{3}AC = \sqrt{3}(3 + \sqrt{3}) = 3\sqrt{3} + 3$.
Therefore, the side lengths of $\triangle ABC$ are $AC = 3+\sqrt{3}$, $AB = 3\sqrt{3}+3$, and $BC = 6+2\sqrt{3}$.

(c) After the described transformation, we obtain the following diagram.



Drop perpendiculars from Q, R and S to D, E and F respectively on BC. Since the circles with centres Q, R and S are tangent to BC, then D, E and F are the points of tangency of these circles to BC.
Thus, $QD = SF = 1$ and $RE = r$.
Join QR, RS, PS, PQ, and PR.
Since we are connecting centres of tangent circles, then $PQ = PS = 2$
and $QR = RS = PR = 1 + r$.
Join QS.
By symmetry, PRE is a straight line (that is, PE passes through R).
Since QS is parallel to BC as in parts (a) and (b), then QS is perpendicular to PR, meeting at Y.



Since $QD = 1$, then $YE = 1$. Since $RE = r$, then $YR = 1 - r$.
Since $QR = 1+r$, $YR = 1-r$ and $\triangle QYR$ is right-angled at Y, then, by the Pythagorean

Theorem,

$$QY^2 = QR^2 - YR^2 = (1+r)^2 - (1-r)^2 = (1+2r+r^2) - (1-2r+r^2) = 4r$$

Since $PR = 1 + r$ and $YR = 1 - r$, then $PY = PR - YR = 2r$.
Since $\triangle PYQ$ is right-angled at Y, then

$$\begin{aligned} PY^2 + YQ^2 &= PQ^2 \\ (2r)^2 + 4r &= 2^2 \\ 4r^2 + 4r &= 4 \\ r^2 + r - 1 &= 0 \end{aligned}$$

By the quadratic formula, $r = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$.
Since $r > 0$, then $r = \dfrac{-1 + \sqrt{5}}{2}$ (which is the reciprocal of the famous "golden ratio").