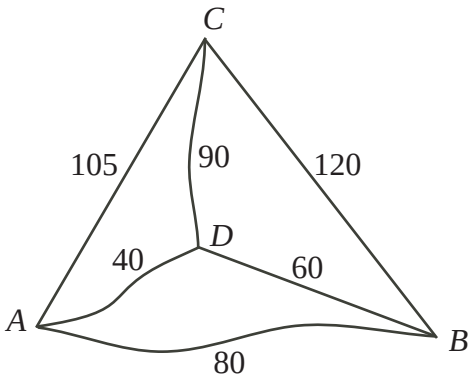
2007 Hypatia Contest (Grade 11)
Wednesday, April 18, 2007

1. The diagram shows four cities A, B, C, and D, with the distances between them in kilometres.



   (a) Penny must travel from A through each of the other cities exactly once and then back to A. An example of her route might be $A \to B \to D \to C \to A$.
   List all routes that Penny could travel.

   (b) Identify one route of the shortest possible length and one of the longest possible length. Explain how you obtained your answer.

   (c) Just before leaving A, Penny learns that
   - she must visit a fifth city E,
   - E is connected directly to each of A, B, C, and D, and
   - E must be the third city she visits.

   Therefore, the trip would be $A \to \_\_ \to \_\_ \to E \to \_\_ \to A$.
   How many different routes are now possible? Explain how you obtained your answer.

   (d) The trip $A \to D \to C \to E \to B \to A$ is 600 km long.
   The trip $A \to C \to D \to E \to B \to A$ is 700 km long.
   The distance from D to E is 225 km.
   What is the distance from C to E? Explain how you obtained your answer.

2. Olayuk has four pails labelled P, Q, R, and S, each containing some marbles. A "legal move" is to take one marble from each of three of the pails and put the marbles into the fourth pail.

   (a) Initially, the pails contain 9, 9, 1, and 5 marbles. Describe a sequence of legal moves that results in 6 marbles in each pail.

   (b) Suppose that the pails initially contain 31, 27, 27, and 7 marbles. After a number of legal moves, each pail contains the same number of marbles.
      i. Describe a sequence of legal moves to obtain the same number of marbles in each pail.
      ii. Explain why at least 8 legal moves are needed to obtain the same number of marbles in each pail.

   (c) Beginning again, the pails contain 10, 8, 11, and 7 marbles. Explain why there is no sequence of legal moves that results in an equal number of marbles in each pail.

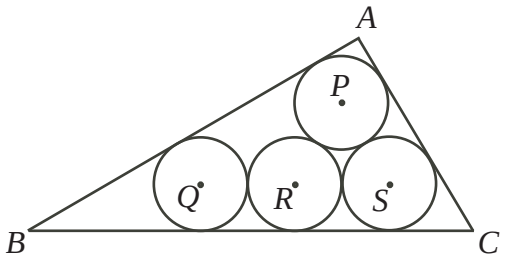
3. Consider the quadratic function $f(x) = x^2 - 4x - 21$.

   (a) Determine all values of x for which $f(x) = 0$ (that is, $x^2 - 4x - 21 = 0$).

   (b) If s and t are different real numbers such that $s^2 - 4s - 21 = t^2 - 4t - 21$ (that is, $f(s) = f(t)$), determine the possible values of $s + t$. Explain how you obtained your answer.

   (c) If a and b are different positive integers such that $(a^2 - 4a - 21) - (b^2 - 4b - 21) = 4$, determine all possible values of a and b. Explain how you obtained your answer.

4. In the diagram, four circles of radius 1 with centres P, Q, R, and S are tangent to one another and to the sides of $\triangle ABC$, as shown.



   (a) Determine the size of each of the angles of $\triangle PQS$. Explain how you obtained your answer.

   (b) Determine the length of each side of $\triangle ABC$. Explain how you obtained your answer.

   (c) The radius of the circle with centre R is decreased so that

   - the circle with centre R remains tangent to BC,
   - the circle with centre R remains tangent to the other three circles, and
   - the circle with centre P becomes tangent to the other three circles.

   This changes the size and shape of $\triangle ABC$. Determine r, the new radius of the circle with centre R.