Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Pascal Contest

(Grade 9)

Wednesday, February 22, 2006

Solutions

1. Calculating each of the numerator and denominator first, $\frac{550+50}{5^2+5} = \frac{600}{25+5} = \frac{600}{30} = 20$.
ANSWER: (E)

2. Calculating under each square root first, $\sqrt{36+64} - \sqrt{25-16} = \sqrt{100} - \sqrt{9} = 10 - 3 = 7$.
ANSWER: (B)

3. The positive whole numbers which divide exactly into 18 are 1, 2, 3, 6, 9, 18, of which there are 6.
ANSWER: (D)

4. Since $A + B = 5$, then $B - 3 + A = (A + B) - 3 = 5 - 3 = 2$.
ANSWER: (A)

5. The volume of the rectangular solid is $2 \times 4 \times 8 = 64$.
If the length of each edge of the cube is s, then the volume of the cube is s^3, which must be equal to 64. Since $s^3 = 64$, then $s = 4$, so the length of each edge of the cube is 4.
ANSWER: (B)

6. Since Ravindra ate $\frac{2}{5}$ of the pizza and Hongshu ate half as much as Ravindra, then Hongshu ate $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$ of the pizza.
After these two had eaten, there was $1 - \frac{2}{5} - \frac{1}{5} = \frac{2}{5}$ of the pizza left.
As a percentage, $\frac{2}{5}$ is equivalent to 40%, so there was 40% of the original pizza left.
ANSWER: (C)

7. Since 1 triangle balances 2 squares, then 2 triangles balance 4 squares.
Since 2 triangles also balance 3 circles, then 3 circles balance 4 squares.
ANSWER: (E)

8. Since the areas of the three squares are 16, 49 and 169, then their side lengths are $\sqrt{16} = 4$, $\sqrt{49} = 7$ and $\sqrt{169} = 13$, respectively.
Thus, the average of their side lengths is $\frac{4+7+13}{3} = 8$.
ANSWER: (A)

9. Since the rectangle has width w, length 8, and perimeter 24, then $2w+2(8) = 24$ or $2w+16 = 24$ or $2w = 8$ or $w = 4$.
Therefore, the ratio of the width to the length is $4 : 8 = 1 : 2$.
ANSWER: (C)

10. *Solution 1*
Looking at the numbers in terms of their digits, then $M4 - 3N = 16$ or $M4 = 3N + 16$.
In order to get a units digit of 4 from $3N + 16$, then N must be an 8.
Thus, $M4 = 38 + 16 = 54$.
Therefore, the digit M is a 5, and so $M + N = 5 + 8 = 13$.

Solution 2
Looking at the numbers in terms of their digits, then $M4 - 3N = 16$.
In order to get a units digit of 6 from $M4 - 3N$, then N must be an 8.
Thus, $M4 - 38 = 16$ or $M4 = 36 + 16 = 54$.
Therefore, the digit M is a 5, and so $M + N = 5 + 8 = 13$.
ANSWER: (D)

11. Evaluating each of the given choices with $x = 9$,

$$\sqrt{9} = 3 \qquad \frac{9}{2} = 4\tfrac{1}{2} \qquad 9 - 5 = 4 \qquad \frac{40}{9} = 4\tfrac{4}{9} \qquad \frac{9^2}{20} = \frac{81}{20} = 4\tfrac{1}{20}$$

Since $\frac{1}{2}$ is larger than either $\frac{4}{9}$ or $\frac{1}{20}$, then the largest of the possibilities when $x = 9$ is $\dfrac{x}{2}$.

Answer: (B)

12. Since the perimeter of the triangle is 36, then $7 + (x + 4) + (2x + 1) = 36$ or $3x + 12 = 36$ or $3x = 24$ or $x = 8$.
Thus, the lengths of the three sides of the triangle are 7, $8+4 = 12$ and $2(8)+1 = 17$, of which the longest is 17.

Answer: (C)

13. *Solution 1*
From the given information, $P + Q = 16$ and $P - Q = 4$.
Adding these two equations, we obtain $P + Q + P - Q = 16 + 4$ or $2P = 20$ or $P = 10$.

Solution 2
The value of P is increased by Q to give 16 and decreased by Q to give 4.
Thus, the difference of 12 between these two answers is twice the value of Q, so $2Q = 12$ whence $Q = 6$.
Since $P + Q = 16$, we have $P + 6 = 16$ or $P = 10$.

Answer: (D)

14. Using a common denominator of 12, we have $\dfrac{6}{12} + \dfrac{8}{12} + \dfrac{9}{12} + \dfrac{n}{12} = \dfrac{24}{12}$ or $\dfrac{23+n}{12} = \dfrac{24}{12}$.
Comparing numerators, we obtain $23 + n = 24$ or $n = 1$.

Answer: (E)

15. *Solution 1*
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes or $1\frac{3}{4}$ hours or $\frac{7}{4}$ hours.
Since Jim drives 84 km in $\frac{7}{4}$ hours at a constant speed, then this speed is $\dfrac{84}{\frac{7}{4}} = 84 \times \dfrac{4}{7} = 48$ km/h.

Solution 2
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes, which is the same as 7 quarters of an hour.
Since he drives 84 km in 7 quarters of an hour, he drives 12 km in 1 quarter of an hour, or 48 km in one hour, so his speed is 48 km/h.

Answer: (E)

16. We make a chart to determine the sum of each possible combination of top faces. In the chart, the numbers across the top are the numbers from the first die and the numbers down the side are the numbers from the second die. For example, the number in the fourth column and fifth row is the sum of the fourth possible result from the first die and the fifth possible result from the second die, or $3 + 5 = 8$.

	2	2	3	3	5	8
2	4	4	5	5	7	10
2	4	4	5	5	7	10
3	5	5	6	6	8	11
3	5	5	6	6	8	11
5	7	7	8	8	10	13
8	10	10	11	11	13	16

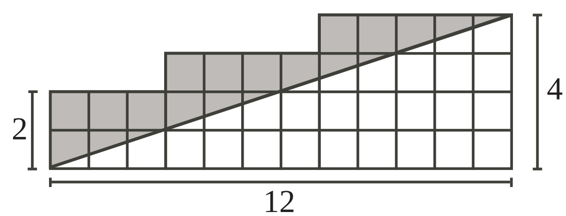
So the possibilities are 4, 5, 6, 7, 8, 10, 11, 13, 16, or nine possibilities in total.
(We could have cut down the size of our table since we didn't have to include both 2's and both 3's either across the top or down the side. As well, we could have also calculated only the numbers on the diagonal and above, since the chart is symmetric.)

ANSWER: (D)

17. Since $\triangle ADE$ is isosceles, then $\angle AED = \angle EAD = 70^\circ$.
Since the angles in $\triangle ADE$ add to 180°, then $\angle ADE = 180^\circ - 2(70^\circ) = 40^\circ$.
Since $\angle DEC = 2(\angle ADE)$, then $\angle DEC = 2(40^\circ) = 80^\circ$.
Since AEB is a straight line, then $\angle CEB = 180^\circ - 80^\circ - 70^\circ = 30^\circ$.
Since $\triangle EBC$ is isosceles, then $\angle ECB = \angle EBC$.
Thus, in $\triangle EBC$, $30^\circ + 2(\angle EBC) = 180^\circ$ or $2(\angle EBC) = 150^\circ$ or $\angle EBC = 75^\circ$.

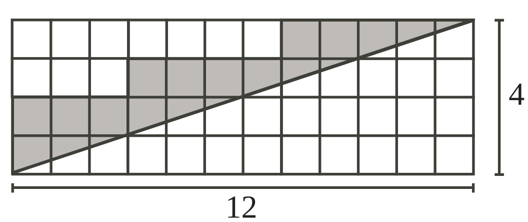
ANSWER: (A)

18. *Solution 1*
The area of the entire grid in the diagram is 38. (We can obtain this either by counting the individual squares, or by dividing the grid into a 2 by 3 rectangle, a 3 by 4 rectangle, and a 4 by 5 rectangle.)



The area of shaded region is equal to the area of the entire grid minus the area of the unshaded triangle, which is right-angled with a base of 12 and a height of 4.
Therefore, the area of the shaded region is $38 - \frac{1}{2}(12)(4) = 38 - 24 = 14$.

Solution 2
First, we "complete the rectangle" by adding more unshaded squares to obtain a 4 by 12 rectangle whose area is $4(12) = 48$.



Note that we added 10 unshaded squares (whose combined area is 10).
The area of the triangle under the line is half of the area of the entire rectangle, or $\frac{1}{2}(48) = 24$.
Thus, the area of the shaded region is the area of the entire rectangle minus the area of the unshaded region, or $48 - 24 - 10 = 14$.

ANSWER: (C)

19. *Solution 1*
Let the ten integers be n, $n+1$, $n+2$, $n+3$, $n+4$, $n+5$, $n+6$, $n+7$, $n+8$, and $n+9$.
Therefore, $S = n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5)+(n+6)+(n+7)+(n+8)+(n+9)$ or $S = 10n + 45$ and $T = 10n$.
Thus, $S - T = (10n + 45) - 10n = 45$.

Solution 2
Since the question implies that the value of $S - T$ must be the same no matter what 10 integers we try, then we calculate $S - T$ for the integers 1 through 10.
In this case, $S = 1+2+3+4+5+6+7+8+9+10 = 55$ and $T = 10(1) = 10$ or $S - T = 45$.

ANSWER: (A)

20. Let w be the width of each of the identical rectangles.
Since $PQ = 3w$, $RS = 2x$ and $PQ = RS$ (because $PQRS$ is a rectangle), then $2x = 3w$, or $w = \frac{2}{3}x$.
Therefore, the area of each of the five identical rectangles is $x\left(\frac{2}{3}x\right) = \frac{2}{3}x^2$.
Since the area of $PQRS$ is 4000 and it is made up of five of these identical smaller rectangles, then $5\left(\frac{2}{3}x^2\right) = 4000$ or $\frac{10}{3}x^2 = 4000$ or $x^2 = 1200$ or $x \approx 34.6$, which, of the possible answers, is closest to 35.

ANSWER: (A)

21. *Solution 1*
Looking at the third row of the table, $(m+8)+(4+n) = 6$ or $m+n+12 = 6$ or $m+n = -6$.
The sum of the nine numbers in the table is
$m+4+m+4+8+n+8+n+m+8+4+n+6 = 3(m+n)+42 = 3(-6)+42 = 24$

Solution 2
Try setting $m = 0$.

Then the table becomes

0	4	4
8	n	$8+n$
8	$4+n$	6

.

From the third row, $8 + (4+n) = 6$ or $n + 12 = 6$ or $n = -6$.

The table thus becomes

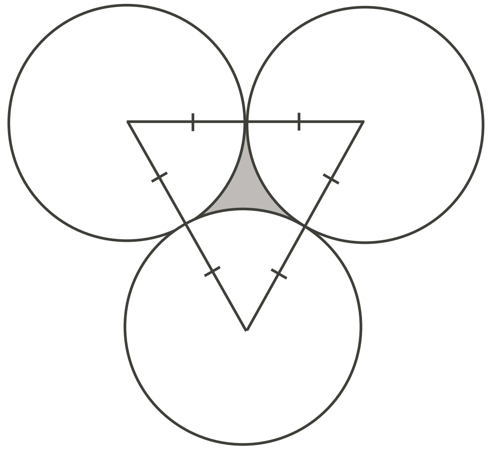
0	4	4
8	-6	2
8	-2	6

.

The sum of the nine numbers in the table is $0+4+4+8+(-6)+2+8+(-2)+6 = 24$.

ANSWER: (E)

22. Join the centre of each circle to the centre of the other two.
Since each circle touches each of the other two, then these line segments pass through the points where the circles touch, and each is of equal length (that is, is equal to twice the length of the radius of one of the circles).



Since each of these line segments have equal length, then the triangle that they form is equilateral, and so each of its angles is equal to 60°.
Now, the perimeter of the shaded region is equal to the sum of the lengths of the three circular arcs which enclose it. Each of these arcs is the arc of one of the circles between the points where this circle touches the other two circles.
Thus, each arc is a 60° arc of one of the circles (since the radii joining either end of each arc to the centre of its circle form an angle of 60°), so is $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$ of the total circumference of the circle, so has length $\frac{1}{6}(36) = 6$.
Therefore, the perimeter of the shaded region is $3(6) = 18$.

Answer: (A)

23. *Solution 1*
Let A be the number of CDs that Anna has, and let B be the number of CDs that Ben has.
If Anna gives 6 CDs to Ben, then Anna would have $A-6$ CDs and Ben would have $B+6$ CDs, so from the given information, $B+6 = 2(A-6)$.
If Anna takes 6 CDs from Ben, then Anna would have $A+6$ CDs and Ben would have $B-6$ CDs, so from the given information, $A+6 = B-6$.
From the first equation, $B = 2A - 18$; from the second equation, $B = A + 12$.
Therefore, $2A - 18 = A + 12$ or $A = 30$, and so $B = A + 12 = 42$.
Thus, the total number of CDs that Ben and Anna have is $30 + 42 = 72$.

Solution 2
Let A be the number of CDs that Anna has.
If Anna receives 6 CDs from Ben, then the two of them would have the same number of CDs.
This tells us that Ben has 12 more CDs than Anne, or that Ben has $A + 12$ CDs.
If Anna gives 6 CDs to Ben, then Anna would have $A - 6$ CDs and Ben would have $A + 18$ CDs.
From the given information, $A + 18 = 2(A - 6)$ or $A + 18 = 2A - 12$ or $A = 30$.
Therefore, Anna has 30 CDs and Ben has $30 + 12 = 42$ CDs, so they have $30 + 42 = 72$ CDs in total.

Answer: (C)

24. *Solution 1*
Suppose that Igor has removed some balls from the bag, and the remaining balls do not satisfy the required condition. What is the maximum number of balls that can remain? In order to

not satisfy the required condition, either there are not 4 balls of any colour (so the maximum number is 9 balls, ie. 3 of each colour) or there are at least 4 balls of one colour, but there are not 3 of either of the other colours.
In this second case, we could have 2 balls of each of two colours, and as many as possible of the third colour. The maximum number of balls of any colour that can be in the bag is 8 (the number of yellow balls with which Igor starts). So the maximum number of balls still in the bag in this case is 12.
Therefore, if Igor removes 8 or more balls, then the remaining balls might not satisfy the required condition.
However, if Igor removes 7 or fewer balls, then the remaining balls will satisfy the required condition, since the maximum number of balls in any case which does not satisfy the condition is 12.
Therefore, the maximum possible value of N is 7.

Solution 2
Since we want to determine the maximum possible value of N, we start with the largest of the answers and rule out answers until we come to the correct answer.
If Igor removed 10 marbles, he might remove 5 red and 5 black marbles, leaving 8 yellow, 2 red, and 0 black marbles, which does not meet the required condition.
Thus, 10 is not the answer.
If Igor removed 9 marbles, he might remove 5 red and 4 black marbles, leaving 8 yellow marbles, 2 red marbles, and 1 black marble, which does not meet the required condition.
Thus, 9 is not the answer.
If Igor removed 8 marbles, he might remove 5 red and 3 black marbles, leaving 8 yellow, 2 red, and 2 black marbles, which does not meet the required condition.
Thus, 8 is not the answer.

Is 7 the answer?
There are $8 + 7 + 5 = 20$ marbles to begin with. If 7 are removed, there are 13 marbles left.
Since there are 13 marbles left, then it is not possible to have 4 or fewer marbles of each of the three colours (otherwise there would be at most 12 marbles). Thus, there are at least 5 marbles of one colour.
Could there be 2 or fewer marbles of each of the other two colours? If so, then since there are 13 marbles in total, there must be at least 9 marbles of the first colour. But there cannot be 9 or more marbles of any colour, as there were at most 8 of each colour to begin with. Therefore, there must be at least 3 of one of the other two colours of marbles.
This tells us that if 7 marbles are removed, there are at least 5 marbles of one colour and 3 of another colour, so choosing $N = 7$ marbles guarantees us the required condition.
Therefore, 7 is the maximum possible value of N.

ANSWER: (B)

25. We will refer to the digits of each of John's and Judith's numbers from the left. Thus, "the first digit" will be the leftmost digit.

If the first digit of John's number is 1, then Judith's number will begin 112.
If the first digit of John's number is 2, then Judith's number will begin 111.
In either case, Judith's number begins with a 1.
Since the first 2187 digits are the same, then John's number begins with a 1.

Since John's number begins with a 1, then Judith's begins 112, so John's begins 112.
Since John's number begins 112, then Judith's begins 112112111, so John's begins 112112111.
Each time we repeat this process, the length of the string which we know will be multiplied by 3.
We continue this process to construct the $2187 = 3^7$ digits of John's number.

We make a table to keep track of this information. We notice that if at one step, the string ends in a 1, then at the next step it will end in a 2, since the 1 becomes 112. Similarly, if at one step, the string ends in a 2, then at the next step, it ends in a 1, since the 2 becomes 111. Also, since each 1 becomes 112 and each 2 becomes 111, then the number of 2's at a given step will be equal to the number of 1's at the previous step. Similarly, the number of 1's at a given step equals 2 times the number of 1's at the previous step plus 3 times the number of 2's at the previous step. (Alternatively, we could determine the total number of 1's by subtracting the number of 2's from the length of the string.)

Step #	Length	# of 1's	# of 2's	Ends in
0	1	1	0	1
1	3	2	1	2
2	9	7	2	1
3	27	20	7	2
4	81	61	20	1
5	243	182	61	2
6	729	547	182	1
7	2187	1640	547	2

How can five consecutive 1's (that is, 11111) be produced in this step 7?
There can never be two consecutive 2's at a given step, since every 2 is the end of one of the blocks and so must be followed by a 1.
Thus, there can never be two consecutive 2's which would produce 111111.
This tells us that 11111 can only be produced by 21 at the previous step.
So the number of occurrences of 11111 at step 7 is equal to the number of occurrences of 21 at step 6. But every 2 at step 6 is followed by a 1 (since the string at step 6 does not end with a 2), so this is equal to the number of 2's at step 6.
Therefore, there are 182 occurrences of 11111 in the 2187 digits of John's number.

Answer: (A)