**Canadian Mathematics Competition**

**An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario**

# 2005 Pascal Contest

**(Grade 9)**

**Wednesday, February 23, 2005**

*Solutions*

1. Calculating each of the numerator and denominator first, $\dfrac{200+10}{20+10} = \dfrac{210}{30} = 7$.

   Answer: (E)

2. Simplifying the terms in pairs, $6a - 5a + 4a - 3a + 2a - a = a + a + a = 3a$.

   Answer: (A)

3. When we substitute $x = 3$, the product becomes $3(2)(1)(0)(-1)$.
   Since one of the factors is 0, the entire product is equal to 0.

   Answer: (C)

4. Of the numbers 2, 3, 4, 5, 6, 7, only the numbers 2, 3, 5, and 7 are prime.
   Since 4 out of the 6 numbers are prime, then the probability of choosing a ball with a prime number is $\frac{4}{6} = \frac{2}{3}$.

   Answer: (D)

5. Calculating from the inside out, $\sqrt{36 \times \sqrt{16}} = \sqrt{36 \times 4} = \sqrt{144} = 12$.

   Answer: (A)

6. When half of the water is removed from the glass, the mass decreases by $1000 - 700 = 300$ g.
   In other words, the mass of half of the original water is 300 g.
   Therefore, the mass of the empty glass equals the mass of the half-full glass, minus the mass of half of the original water, or $700 - 300 = 400$ g.

   Answer: (D)

7. *Solution 1*
   Since $\frac{1}{3}x = 12$, then $x = 3 \times 12 = 36$, so $\frac{1}{4}x = \frac{1}{4}(36) = 9$.

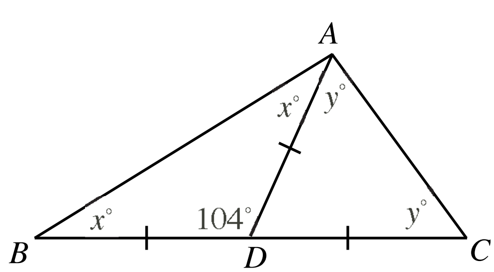
   *Solution 2*
   Since $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$, then $\frac{1}{4}x = \frac{3}{4} \times (\frac{1}{3}x) = \frac{3}{4}(12) = 9$.

   Answer: (C)

8. We calculate the square of each number: $(-5)^2 = 25$, $(\frac{3}{2})^2 = \frac{9}{4}$, $2^2 = 4$, $(\frac{3}{5})^2 = \frac{9}{25}$, and $8^2 = 64$.
   We can quickly tell that each of $-5$, 2 and 8 is less than its square.
   Since $\frac{3}{2} = 1\frac{1}{2}$ and $\frac{9}{4} = 2\frac{1}{4}$, then $\frac{3}{2}$ is also less than its square.
   However, $\frac{3}{5} = \frac{15}{25}$ is larger than its square $\frac{9}{25}$.

   Answer: (D)

9. *Solution 1*
   Since $\triangle BDA$ is isosceles, $\angle BAD = \angle ABD = x^\circ$.
   Since $\triangle CDA$ is isosceles, $\angle CAD = \angle ACD = y^\circ$.

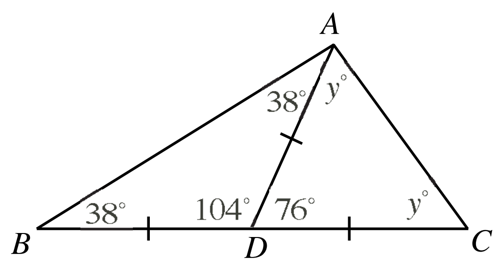


Therefore, $\angle BAC = (x+y)^\circ$.
Since the sum of the angles in $\triangle ABC$ is $180^\circ$, then

$$\begin{aligned} x^\circ + y^\circ + (x+y)^\circ &= 180^\circ \\ (2x+2y)^\circ &= 180^\circ \\ 2x+2y &= 180 \\ x+y &= 90 \end{aligned}$$

Therefore, $x+y=90$.

*Solution 2*
Since $\triangle BDA$ is isosceles, $\angle BAD = \angle ABD = x^\circ$.
Therefore, looking at the sum of the angles in $\triangle ABD$, we have $x^\circ + x^\circ + 104^\circ = 180^\circ$ or $2x + 104 = 180$ or $2x = 76$ or $x = 38$.
Since $\angle BDA$ and $\angle ADC$ are supplementary, then $\angle ADC = 180^\circ - 104^\circ = 76^\circ$.
Since $\triangle CDA$ is isosceles, $\angle CAD = \angle ACD = y^\circ$.




Therefore, looking at the sum of the angles in $\triangle CDA$, we have $y^\circ + y^\circ + 76^\circ = 180^\circ$ or $2y + 76 = 180$ or $2y = 104$ or $y = 52$.
Therefore, $x + y = 38 + 52 = 90$.

Answer: (D)

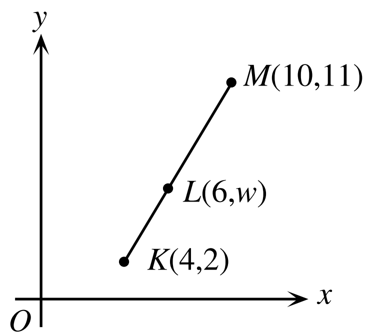
10. The third term in the sequence is, by definition, the average of the first two terms, namely 32 and 8, or $\frac{1}{2}(32+8) = \frac{1}{2}(40) = 20$.
The fourth term is the average of the second and third terms, or $\frac{1}{2}(8+20) = 14$.
The fifth term is the average of the third and fourth terms, or $\frac{1}{2}(20+14) = 17$.
Therefore, $x = 17$.

Answer: (A)

11. If $a$ and $b$ are positive integers with $a \times b = 13$, then since 13 is a prime number, we must have $a$ and $b$ equal to 1 and 13, or 13 and 1, respectively.
If $b = 1$, then since $b \times c = 52$, we must have $c = 52$. But then we could not have $c \times a = 4$. So $b$ cannot be 1.
Thus, $b = 13$, so $a = 1$, and $c = 4$ (which we can get either from knowing $b = 13$ and $b \times c = 52$, or from knowing $a = 1$ and $c \times a = 4$.
Therefore, $a \times b \times c = 1 \times 13 \times 4 = 52$.

Answer: (E)

12. *Solution 1*
To get from $K$ to $M$, we move 6 units to the right and 9 units up.




Since $L$ lies on the line segment $KM$ and to get from $K$ to $L$ we move $2 = \frac{1}{3} \times 6$ units to the right, then we must move $\frac{1}{3} \times 9 = 3$ units up.
Thus, $w = 2 + 3 = 5$.

*Solution 2*
The slope of line segment $KM$ is $\dfrac{11-2}{10-4} = \dfrac{9}{6} = \dfrac{3}{2}$.
Since $L$ lies on line segment $KM$, then the slope of line segment $KL$ will also be $\dfrac{3}{2}$.
Therefore, $\dfrac{w-2}{6-4} = \dfrac{3}{2}$ or $\dfrac{w-2}{2} = \dfrac{3}{2}$, so $w - 2 = 3$ or $w = 5$.

Answer: (B)

13. *Solution 1*
Each unit cube has three exposed faces on the larger cube, and three faces hidden.
Therefore, when the faces of the larger cube are painted, 3 of the 6 faces (or $\frac{1}{2}$ of the surface area) of each small cube are painted, so overall exactly $\frac{1}{2}$ of the surface of area of the small cubes is painted.

*Solution 2*
Since the bigger cube is 2 by 2 by 2, then it has six 2 by 2 faces, for a total surface area of $6 \times 2 \times 2 = 24$ square units. All of this surface area will be painted.
Each of the unit cubes has a surface area of 6 square units (since each has six 1 by 1 faces), so the total surface area of the 8 unit cubes is $8 \times 6 = 48$ square units.
Of this 48 square units, 24 square units is red, for a fraction of $\frac{24}{48} = \frac{1}{2}$.

Answer: (C)

14. Every integer between 2005 and 3000 has four digits.
Since palindromes are integers whose digits are the same when read forwards and backwards, then a four-digit palindrome must be of the form $xyyx$ where $x$ and $y$ are digits.
Since 3000 is not a palindrome, then the first digit of each of these palindromes must be 2, ie. it must be of the form $2yy2$.
Since each palindrome is at least 2005 and at most 3000, then $y$ can take any value from 1 to 9.
Therefore, there are 9 such palindromes: 2112, 2222, 2332, 2442, 2552, 2662, 2772, 2882, and 2992.

Answer: (C)

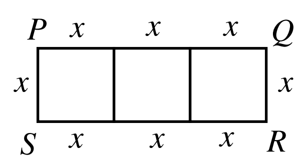
15. When 14 is divided by $n$, the remainder is 2, so $n$ must divide evenly into $14 - 2 = 12$. Therefore, $n$ could be 1, 2, 3, 4, 6, or 12.
    But the remainder has to be *less than* the number we are dividing by, so $n$ cannot be 1 or 2.
    Thus, $n$ can be 3, 4, 6 or 12, so there are 4 possible values of $n$.

    ANSWER: (D)

16. To get the largest possible number by using the digits 1, 2, 5, 6, and 9 exactly once, we must choose the largest digit to be the first digit, the next largest for the next digit, and so on. Thus, the largest possible number is 96 521.
    Also, using this reasoning, there are only two numbers that use these digits which are at least 96 500, namely 96 521 and 96 512.
    Therefore, 96 512 must be the largest even number that uses each of these digits exactly once.
    Reversing our logic, the smallest possible number formed using these digits is 12 569, and the only other such number smaller than 12 600 is 12 596, so this must be the smallest even number formed using these digits.
    Calculating the difference, we get $96\,512 - 12\,596 = 83\,916$.

    ANSWER: (A)

17. Let the side length of each of the squares be $x$.




Then the perimeter of $PQRS$ equals $8x$, so $8x = 120$ cm or $x = 15$ cm.
Since $PQRS$ is made up of three squares of side length 15 cm, then its area is $3(15\text{ cm})^2$ or $3(225)\text{ cm}^2 = 675\text{ cm}^2$.

ANSWER: (B)

18. First, $2005^2 = 4020025$, so the last two digits of $2005^2$ are 25.
    We need to look at $2005^5$, but since we only need the final two digits, we don't actually have to calculate this number entirely.
    Consider $2005^3 = 2005^2 \times 2005 = 4020025 \times 2005$. When we carry out this multiplication, the last two digits of the product will only depend on the last two digits of the each of the two numbers being multiplied (try this by hand!), so the last two digits of $2005^3$ are the same as the last two digits of $25 \times 5 = 125$, ie. are 25.
    Similarly, to calculate $2005^4$, we multiply $2005^3$ (which ends in 25) by 2005, so by the same reasoning $2005^4$ ends in 25.
    Similarly, $2005^5$ ends in 25.
    Therefore, $2005^2$ and $2005^5$ both end in 25.
    Also, $2005^0 = 1$, so the expression overall is equal to $\ldots 25 + 1 + 1 + \ldots 25 = \ldots 52$.
    Therefore, the final two digits are 52.

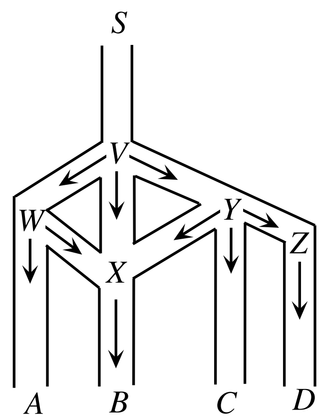
    ANSWER: (A)

19. The easiest way to count these numbers is to list them by considering their hundreds digit.
    Hundreds digit of 1: No such numbers, since tens digit must be 0, which leaves no option for units digit.
    Hundreds digit of 2: Only one possible number, namely 210.

Hundreds digit of 3: Here the tens digit can be 2 or 1, giving numbers 321, 320, 310.
Hundreds digit of 4: Here the tens digit can be 3, 2 or 1, giving numbers 432, 431, 430, 421, 420, 410.
Therefore, there are 10 such numbers in total.

Answer: (B)

20. We label the five junctions as $V$, $W$, $X$, $Y$, and $Z$.



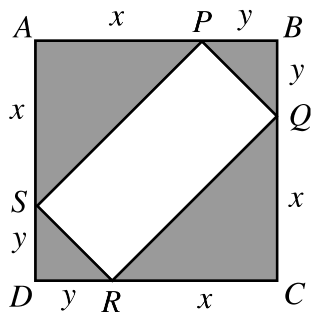

From the arrows which Harry can follow, we see that in order to get to $B$, he must go through $X$ (and from $X$, he has to go to $B$). So we calculate the probability that he gets to $X$.
To get to $X$, Harry can go $S$ to $V$ to $W$ to $X$, or $S$ to $V$ to $Y$ to $X$, or $S$ to $V$ to $X$ directly.
At $V$, the probability that Harry goes down any of the three paths (that is, towards $W$, $X$ or $Y$) is $\frac{1}{3}$.
So the probability that Harry goes directly from $V$ to $X$ to $\frac{1}{3}$.
At $W$, the probability that Harry turns to $X$ is $\frac{1}{2}$, so the probability that he goes from $V$ to $W$ to $X$ is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.
At $Y$, the probability that Harry turns to $X$ is $\frac{1}{3}$, so the probability that he goes from $V$ to $Y$ to $X$ is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.
Therefore, the probability that Harry gets to $X$ (and thus to $B$) is $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{6+3+2}{18} = \frac{11}{18}$.

Answer: (C)

21. We try each of the five possibilities.
If $m : n = 9 : 1$, then we set $m = 9x$ and $n = x$, so $9x + x = 300$ or $10x = 300$ or $x = 30$, so $m = 9(30) = 270$ and $n = 30$.
If $m : n = 17 : 8$, then we set $m = 17x$ and $n = 8x$, so $17x + 8x = 300$ or $25x = 300$ or $x = 12$, so $m = 17(12) = 204$ and $n = 8(12) = 96$.
If $m : n = 5 : 3$, then we set $m = 5x$ and $n = 3x$, so $5x + 3x = 300$ or $8x = 300$ or $x = \frac{75}{2}$, so $m = 5(\frac{75}{2}) = \frac{375}{2}$ and $n = 3(\frac{75}{2}) = \frac{225}{2}$.
If $m : n = 4 : 1$, then we set $m = 4x$ and $n = x$, so $4x + x = 300$ or $5x = 300$ or $x = 60$, so $m = 4(60) = 240$ and $n = 60$.
If $m : n = 3 : 2$, then we set $m = 3x$ and $n = 2x$, so $3x + 2x = 300$ or $5x = 300$ or $x = 60$, so $m = 3(60) = 180$ and $n = 2(60) = 120$.
The only one of the possibilities for which $m$ and $n$ are integers, each greater than 100, is $m : n = 3 : 2$.

Answer: (E)

22. Let $AS = x$ and $SD = y$.
Since $\triangle SAP$ and $\triangle SDR$ are isosceles, then $AP = x$ and $DR = y$.
Since there are two pairs of identical triangles, then $BP = BQ = y$ and $CQ = CR = x$.




$\triangle SDR$ is right-angled (since $ABCD$ is a square) and isosceles, so its area (and hence the area of $\triangle BPQ$) is $\frac{1}{2}y^2$.
Similarly, the area of each of $\triangle SAP$ and $\triangle QCR$ is $\frac{1}{2}x^2$.
Therefore, the total area of the four triangles is $2(\frac{1}{2}x^2) + 2(\frac{1}{2}y^2) = x^2 + y^2$, so $x^2 + y^2 = 200$.
Now, by the Pythagorean Theorem, used first in $\triangle PRS$, then in $\triangle SAP$ and $\triangle SDR$,

$$\begin{aligned} PR^2 &= PS^2 + SR^2 \\ &= (SA^2 + AP^2) + (SD^2 + DR^2) \\ &= 2x^2 + 2y^2 \\ &= 2(200) \\ &= 400 \end{aligned}$$

so $PR = 20$ m.

Answer: (B)

23. From the centre 2, there are 8 possible 0s to which we can move initially: 4 "side" 0s

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | 0 | | |
| | 0 | 2 | 0 | |
| | | 0 | | |
| | | | | |

and 4 "corner" 0s.

| | | | | |
|---|---|---|---|---|
| | | | | |
| | 0 | | 0 | |
| | | 2 | | |
| | 0 | | 0 | |
| | | | | |

From a side 0, there are 4 possible 0s to which we can move: 2 side 0s and 2 corner 0s

| | | | | |
|---|---|---|---|---|
| | | | | |
| | 0 | 0 | 0 | |
| | 0 | | 0 | |
| | | | | |
| | | | | |

From a corner 0, there are 2 possible 0s to which we can move: 2 side 0s.

| | | | | |
|---|---|---|---|---|
| | | | | |
| | 0 | 0 | | |
| | 0 | | | |
| | | | | |
| | | | | |

From a side 0, there are 3 possible 5s to which we can move.

| | | | | |
|---|---|---|---|---|
| | | | | |
| 5 | | | | |
| 5 | 0 | | | |
| 5 | | | | |
| | | | | |

From a corner 0, there are 5 possible 5s to which we can move.

| | | | | |
|---|---|---|---|---|
| 5 | 5 | 5 | | |
| 5 | 0 | | | |
| 5 | | | | |
| | | | | |
| | | | | |

So the three possible combinations of 0s are "side-side", "side-corner" and "corner-side".
For the combination "side-side", there is a total of $4 \times 2 \times 3 = 24$ paths (because there are initially 4 side 0s that can be moved to, and from *each* of these, there are 2 side 0s that can be moved to, and from *each* of those, there are 3 5s that can be moved to).
For the combination "side-corner", there is a total of $4 \times 2 \times 5 = 40$ paths.
For the combination "corner-side", there is a total of $4 \times 2 \times 3 = 24$ paths.
Therefore, in total there are $24 + 40 + 24 = 88$ paths that can be followed to form 2005.

ANSWER: (E)

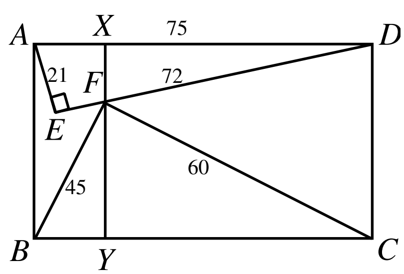
24. Since we have an increasing sequence of integers, the 1000th term will be at least 1000.
We start by determining the number of perfect powers less than or equal to 1000. (This will tell us how many integers less than 1000 are "skipped" by the sequence.)
Let us do this by making a list of all perfect powers less than 1100 (we will need to know the locations of some perfect powers bigger than 1000 anyways):

Perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089
Perfect cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000
Perfect 4th powers: 1, 16, 81, 256, 625
Perfect 5th powers: 1, 32, 243, 1024
Perfect 6th powers: 1, 64, 729
Perfect 7th powers: 1, 128
Perfect 8th powers: 1, 256
Perfect 9th powers: 1, 512
Perfect 10th powers: 1, 1024

In these lists, there are 41 distinct perfect powers less than or equal to 1000.
Thus, there are 959 positive integers less than or equal to 1000 which are not perfect powers.
Therefore, 999 will be the 959th term in the sequence. (This is because 1000 itself is actually a perfect power; if it wasn't, 1000 would be the 959th term.)
Thus, 1001 will be the 960th term.
The next perfect power larger than 1000 is 1024.
Thus, 1023 will be the 982nd term and 1025 will be the 983rd term.
The next perfect power larger than 1024 is 1089.
Therefore, the 1042 will be the 1000th term.
The sum of the squares of the digits of 1042 is $1^2 + 0^2 + 4^2 + 2^2 = 21$.

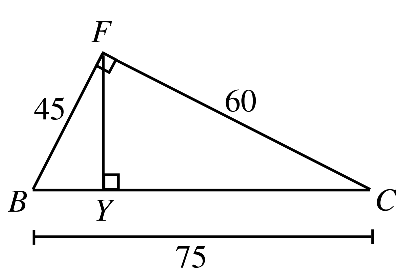
Answer: (E)

25. By the Pythagorean Theorem in $\triangle AED$, $AD^2 = AE^2 + ED^2 = 21^2 + 72^2 = 5625$, so $AD = 75$.
Since $ABCD$ is a rectangle, $BC = AD = 75$. Also, by the Pythagorean Theorem in $\triangle BFC$, $FC^2 = BC^2 - BF^2 = 75^2 - 45^2 = 3600$, so $FC = 60$.
Draw a line through $F$ parallel to $AB$, meeting $AD$ at $X$ and $BC$ at $Y$.
To determine the length of $AB$, we can find the lengths of $FY$ and $FX$.




Step 1: Calculate the length of $FY$

The easiest method to do this is to calculate the area of $\triangle BFC$ in two different ways.
We know that $\triangle BFC$ is right-angled at $F$, so its area is equal to $\frac{1}{2}(BF)(FC)$ or $\frac{1}{2}(45)(60) = 1350$.
Also, we can think of $FY$ as the height of $\triangle BFC$, so its area is equal to $\frac{1}{2}(FY)(BC)$ or $\frac{1}{2}(FY)(75)$.



Therefore, $\frac{1}{2}(FY)(75) = 1350$, so $FY = 36$.

(We could have also approached this by letting $FY = h$, $BY = x$ and so $YC = 75 - x$. We could have then used the Pythagorean Theorem twice in the two little triangles to create two equations in two unknowns.)

Since $FY = 36$, then by the Pythagorean Theorem,

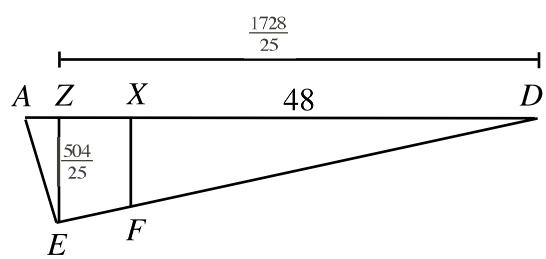
$$BY^2 = BF^2 - FY^2 = 45^2 - 36^2 = 729$$

so $BY = 27$.
Thus, $YC = BC - BY = 48$.

Step 2: Calculate the length of $FX$

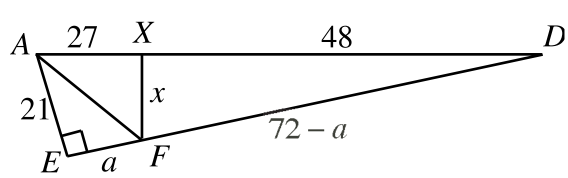
*Method 1 – Similar triangles*
Since $\triangle AED$ and $\triangle FXD$ are right-angled at $E$ and $X$ respectively and share a common angle $D$, then they are similar.
Since $YC = 48$, then $XD = 48$.
Since $\triangle AED$ and $\triangle FXD$ are similar, then $\dfrac{FX}{XD} = \dfrac{AE}{ED}$ or $\dfrac{FX}{48} = \dfrac{21}{72}$ so $FX = 14$.

*Method 2 – Mimicking Step 1*
Drop a perpendicular from $E$ to $AD$, meeting $AD$ at $Z$.




We can use exactly the same argument from Step 1 to calculate that $EZ = \frac{504}{25}$ and that $ZD = \frac{1728}{25}$.
Since $DF$ is a straight line, then the ratio $FX : EZ$ equals the ratio $DX : DZ$, ie. $\dfrac{FX}{\frac{504}{25}} = \dfrac{48}{\frac{1728}{25}}$ or $\dfrac{FX}{504} = \dfrac{48}{1728}$ or $FX = 14$.

*Method 3 – Areas*
Join $A$ to $F$. Let $FX = x$ and $EF = a$. Then $FD = 72 - a$.




Since $AE = 21$ and $ED = 72$, then the area of $\triangle AED$ is $\frac{1}{2}(21)(72) = 756$.
Now, the area of $\triangle AED$ is equal to the sum of the areas of $\triangle AEF$ and $\triangle AFD$, or

$$756 = \frac{1}{2}(21)(a) + \frac{1}{2}(75)(x)$$

so $21a + 75x = 1512$ or $a + \frac{25}{7}x = 72$.
Now in $\triangle FXD$, we have $FX = x$, $GD = 48$ and $FD = 72 - a$.

By the Pythagorean Theorem, $x^2 + 48^2 = (72 - a)^2 = (\frac{25}{7}x)^2 = \frac{625}{49}x^2$.
Therefore, $48^2 = \frac{576}{49}x^2$, or $x = 14$.

Therefore, $AB = XY = FX + FY = 36 + 14 = 50$.

ANSWER: (A)