Canadian
Mathematics
Competition
*An activity of the Centre for Education
in Mathematics and Computing,
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2005 Hypatia Contest

Wednesday, April 20, 2005

Solutions

1. (a) By definition, $2 \diamond 3 = 2^2 - 4(3) = 4 - 12 = -8$.

(b) By definition, $k \diamond 2 = k^2 - 4(2) = k^2 - 8$.
By definition, $2 \diamond k = 2^2 - 4(k) = 4 - 4k$.
So we want to solve

$$\begin{aligned} k^2 - 8 &= 4 - 4k \\ k^2 + 4k - 12 &= 0 \\ (k+6)(k-2) &= 0 \end{aligned}$$

so $k = -6$ or $k = 2$.
Checking, $(-6) \diamond 2 = (-6)^2 - 4(2) = 28$, $2 \diamond (-6) = 2^2 - 4(-6) = 28$, so $k = -6$ works.
Also, if $k = 2$, then $2 \diamond 2 = 2 \diamond 2$ so $k = 2$ works as well.

(c) Since $3 \diamond x = y$, then $3^2 - 4x = y$ or $9 - 4x = y$.
Since $2 \diamond y = 8x$, then $2^2 - 4y = 8x$ or $4 - 4y = 8x$.
We now have a system of two equations in two unknowns.
Since $4 - 4y = 8x$ and $y = 9 - 4x$, then

$$\begin{aligned} 4 - 4(9 - 4x) &= 8x \\ 4 - 36 + 16x &= 8x \\ 8x &= 32 \\ x &= 4 \end{aligned}$$

Since $x = 4$, then $y = 9 - 4(4) = -7$.
(We could have solved this system of equations in several different ways instead.)
Checking, $3 \diamond x = 3 \diamond 4 = 3^2 - 4(4) = -7 = y$ and $2 \diamond y = 2 \diamond (-7) = 2^2 - 4(-7) = 32 = 8x$, so $x = 4$, $y = -7$ is indeed the solution.

2. (a) Since 3, then 1, then 4 toothpicks have been removed from the initial pile of 11 toothpicks, there are now 3 toothpicks remaining.
Since players have removed 1, 3 and 4 toothpicks on turns already, then Chris can only remove 2 or 5 toothpicks now on his turn, because of rules 2 and 3.
Since there are only 3 toothpicks remaining, Chris must remove 2 toothpicks.
This leaves 1 toothpick in the pile, and the only possible move that Gwen can now make is to remove 5 toothpicks, which is impossible.
Therefore, Gwen cannot make her turn.
Since Chris was the last player able to move, then Chris wins.

(b) After Gwen has removed 5 toothpicks, there are 5 remaining and Chris can remove 1, 2, 3, or 4 on his turn.
If Chris removes 1, there are 4 remaining and Gwen can remove all of them (since no one has yet removed 4 toothpicks on a turn). This empties the pile, so Gwen wins.
If Chris removes 2, there are 3 remaining and Gwen can remove all of them (since no one has yet removed 3 toothpicks on a turn). This empties the pile, so Gwen wins.
If Chris removes 3, there are 2 remaining and Gwen can remove all of them (since no one has yet removed 2 toothpicks on a turn). This empties the pile, so Gwen wins.
If Chris removes 4, there is 1 remaining and Gwen can remove all of them (since no one has yet removed 1 toothpick on a turn). This empties the pile, so Gwen wins.
Therefore, no matter what Chris removes, Gwen can always win the game.

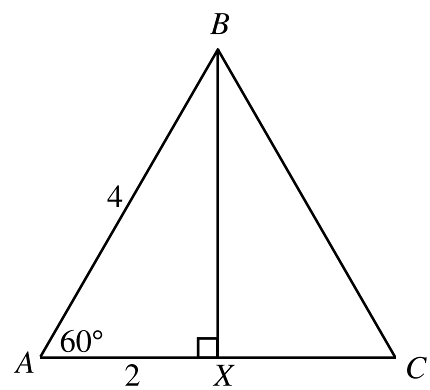
(c) After Gwen has removed 2 toothpicks, there are 7 toothpicks remaining, and Chris can take 1, 3, 4, or 5 on his turn.
If Chris removes 5 toothpicks, there are 2 remaining and Gwen can remove 1, 3 or 4. Thus, Gwen must remove 1, leaving 1 toothpick and Chris can remove 3 or 4. He is unable to make his turn, so Gwen wins. So Chris should not remove 5 toothpicks.

If Chris removes 3 or 4 toothpicks, there are 4 or 3 toothpicks remaning, and Gwen can remove all of them (since in either case that number of toothpicks hasn't yet been removed on a turn), so Gwen wins. So Chris should not remove 3 or 4 toothpicks. (If Gwen removed 1 toothpick instead of 4 or 3 toothpicks, she would still be guaranteed to win, since Chris would be unable to go again. Why?)

If Chris removes 1 toothpick, there are 6 remaining and Gwen can remove 3, 4 or 5. If Gwen now removes 5 toothpicks, there is 1 remaining, and Chris is unable to make his move, since he can now only remove 3 or 4 toothtpicks. So Gwen wins. Similarly, if Gwen had removed 4 toothpicks, there would be 2 remaining and Chris cannot remove 1 or 2 since these numbers have already been used, so Gwen wins. If Gwen had removed 3 toothpicks, there would be 3 remaining and Chris cannot remove 1, 2 or 3, since these numbers have already been used, so Gwen wins.

Thus, regardless of what Chris does on his turn, Gwen will win.

3. (a) *Solution 1*
Drop a perpendicular from B to X on AC.
Since $\triangle ABC$ is equilateral, then $AB = CB$, so X will be the midpoint of AC, so $AX = 2$.



By the Pythagorean Theorem, $BX = \sqrt{AB^2 - AX^2} = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(BX) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.

Solution 2
Drop a perpendicular from B to X on AC.
Since $\triangle ABC$ is equilateral, then $AB = CB$, so X will be the midpoint of AC, so $AX = 2$.
Since $\angle BAX = 60^\circ$ and BX is perpendicular to AX, then $\triangle BAX$ is a 30°-60°-90° triangle, so $BX = \sqrt{3}AX = 2\sqrt{3}$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(BX) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.

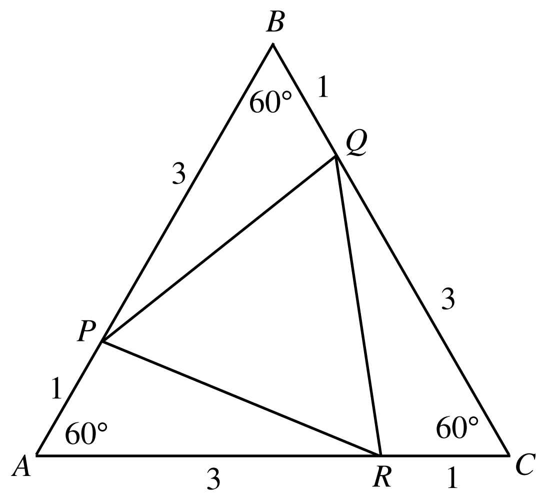
Solution 3
Drop a perpendicular from B to X on AC.
Since $\triangle ABC$ is equilateral, then $AB = CB$, so X will be the midpoint of AC, so $AX = 2$.

Since $\angle BAX = 60^\circ$, then $BX = BA\sin(60^\circ) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(BX) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.

Solution 4
The area of $\triangle ABC$ is given by the formula

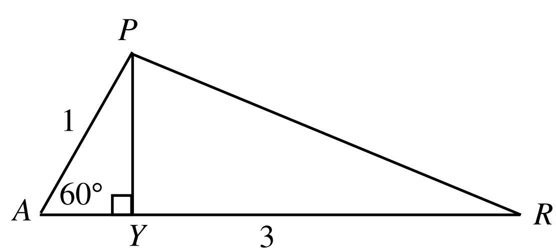
$$\frac{1}{2}(AB)(AC)\sin(\angle BAC) = \frac{1}{2}(4)(4)\sin(60^\circ) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$$

(b) Since $AP = BQ = CR = 1$, then $AR = BP = CQ = 3$, since the side length of $\triangle ABC$ is 4.



Since $AP = BQ = CR = 1$, $PB = QC = RA = 3$ and $\angle RAP = \angle BPQ = \angle QCR = 60^\circ$, then $\triangle RAP$, $\triangle PBQ$ and $\triangle QCR$ are all congruent (by side-angle-side). Therefore, the areas of $\triangle PBQ$, $\triangle RAP$ and $\triangle QCR$ will all be equal.
Finding the area of any of these three triangles will give us the area of all three, so we determine the area of $\triangle RAP$, because this is easiest to visualize.

Method #1
Drop a perpendicular from P to Y on AR.



Then, the area of $\triangle RAP$ is equal to $\frac{1}{2}(AR)(PY) = \frac{1}{2}(3)(PY) = \frac{3}{2}(PY)$, so we need to find the length of PY.
Since $\angle RAP = 60^\circ$, then $PY = AP\sin(60^\circ) = 1\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$.
Therefore, the area of $\triangle RAP$ is $\frac{3}{2}(PY) = \frac{3\sqrt{3}}{4}$.

Method #2
The area of $\triangle RAP$ is $\frac{1}{2}(RA)(AP)\sin(\angle RAP) = \frac{1}{2}(3)(1)\sin(60^\circ) = \frac{3\sqrt{3}}{4}$.

Using either method, we obtain that the area of $\triangle PBQ$ is $\frac{3\sqrt{3}}{4}$.

Lastly, we must determine the area of $\triangle PQR$.
Method #1
To do this, we can subtract the combined areas of $\triangle PBQ$, $\triangle RAP$ and $\triangle QCR$ from the area of the large triangle, $\triangle ABC$.
But the areas of these three triangles are equal (as stated above), and we found the area of $\triangle ABC$ in part (a).
Therefore, the area of $\triangle PQR$ is $4\sqrt{3} - 3\left(\frac{3\sqrt{3}}{4}\right) = \frac{16\sqrt{3}}{4} - \frac{9\sqrt{3}}{4} = \frac{7\sqrt{3}}{4}$.

Method #2
Since $\triangle RAP$, $\triangle PBQ$ and $\triangle QCR$ are all congruent, then $PQ = QR = RP$, so $\triangle PQR$ is equilateral.
So, if we can calculate the side length of $\triangle PQR$, then we can use a similar method to any of the methods from (a) to calculate the area of $\triangle PQR$.
Using the cosine law in $\triangle RAP$, we can find PR:

$$\begin{aligned} PR^2 &= PA^2 + AR^2 - 2(PA)(AR)\cos(\angle PAR) \\ PR^2 &= 1^2 + 3^2 - 2(1)(3)\cos(60^\circ) \\ PR^2 &= 10 - 6\left(\frac{1}{2}\right) \\ PR^2 &= 7 \end{aligned}$$

so $PR = \sqrt{7}$.
We can then use any of the methods from (a) to determine that the area of $\triangle PQR$ is $\frac{7\sqrt{3}}{4}$.

4. (a) *Solution 1*
Since the middle number has to be the largest of the three numbers in the triple, then the only possibilities for b are 3, 4 and 5.
If $b = 3$, then a and c can only be 1 and 2 or 2 and 1, a total of 2 possible triples.
If $b = 4$, then a and c can be 1 and 2, 1 and 3, 2 and 3, or their reverses, a total of 6 possible triples.
If $b = 5$, then a and c can be 1 and 2, 1 and 3, 1 and 4, 2 and 3, 2 and 4, 3 and 4, or their reverses, a total of 12 possible triples.
Thus, in total there are 20 possible triples.

Solution 2
This solution uses the combinatorial notation $\binom{n}{r}$ and $n!$.
First, we choose three different numbers from the set $\{1, 2, 3, 4, 5\}$.
There are $\binom{5}{3} = 10$ ways of doing this.
From these three numbers, to form a triple (a, b, c) with $a < b$ and $b > c$, the middle number b must be the largest, so there is no choice as to what to put in the middle position.
With the two remaining numbers we can put them in the first and last position in either order (ie. two possibilities).
Therefore, each choice of 3 different numbers gives us two possible triples, so the total number of possible triples is $10 \times 2 = 20$.

(To be totally rigorous, we should also note that we can obtain every such triple in this way, and that we don't get any overlap, since we're choosing three different numbers always, and we can't have equal triples coming from two different choices of three numbers.)

(b) *Solution 1*

Each arrangement of $\{1, 2, 3, 4, 5, 6\}$ has one number in each of six positions.
If 254 occurs as a block in the arrangement, then the arrangement must be of one of the forms $254xyz$, $x254yz$ $xy254z$, or $xyz254$, where x, y and z are 1, 3 and 6 in some order.
With each of these 4 forms, there are 6 ways in which the numbers 1, 3 and 6 can fill the three remaining places – either 1,3,6 or 1,6,3 or 3,1,6 or 3,6,1 or 6,1,3 or 6,3,1.
Therefore, there are $4 \times 6 = 24$ arrangements containing 254 consecutively in that order.

Solution 2

We treat 254 as a single block and call it B, say.
Then the arrangements which we are counting correspond to the arrangements of $\{1, 2, 3, B\}$.
There are $4! = 24$ arrangements of the 4 element set $\{1, 2, 3, B\}$ (since there are four possibilities for the first element of the arrangement, and for each of these there are three possibilities for the second element, and so on).
Therefore, there are 24 arrangements containing 254 consecutively in that order.

(c) *Solution 1*

To determine the average number of local peaks in all of the arrangements, we count the total number of local peaks in all 40 320 arrangements and then divide by this total number of arrangements.
To count the total number of local peaks, instead of looking at the arrangements and counting the number of local peaks in the arrangements, we look at the possible local peaks and count the number of arrangements in which each occurs.
In an arrangement of $\{1, 2, 3, 4, 5, 6, 7, 8\}$, a local peak is a sequence of three numbers abc inside the arrangement where $a < b$ and $b > c$.
How many such sequences of three numbers are there?
This is an extension of part (a). Using either technique from (a), we can determine that the total number of such sequences is 112.
Now fix one of these 112 sequences abc. In how many of the 40 320 arrangements does this sequence occur as a block?
This is an extension of part (b). Using either technique from (b), we can determine that the total number of such arrangements is $6! = 720$.
Thus, each of the 112 possible local peak sequences occurs in 720 arrangements, so there are a total of $112 \times 720 = 80\,640$ local peaks in all possible arrangements.
(We have indeed counted all such local peaks, since every local peak occurs as a subsequence of three numbers in this way.)
Therefore, the average number of local peaks in the 40 320 arrangements is $\dfrac{80640}{40320} = 2$.

Solution 2

In an arrangement of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, a local peak involves three consecutive positions, so can occur in one of six places – in positions 1 to 3, 2 to 4, 3 to 5, 4 to 6, 5 to 7, or 6 to 8.
Let's focus on one of these places, say positions 1 to 3. Our arguments will apply equally to all such places.
What fraction of all of the arrangements will have local peak in this position?
Choose three numbers a, b, c from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, say with $a < b < c$.

There are six possible ways to arrange these three numbers: abc, acb, bac, bca, cab, cba.
Of these six possible ways, two will give a local peak: acb and bca (from the condition that $a < b < c$). So $\frac{1}{3}$ of the possible ways to arrange a, b, c give a local peak.
Consider all of the arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose first three numbers are a, b, c in some order.
Since the same total number of arrangements begin with abc as begin with acb as begin with bac as begin with any of the six ways of ordering a, b and c, then exactly $\frac{1}{3}$ of the arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ starting with a, b, c in some order have a local peak across positions 1 to 3.
Since the number of arrangements with any fixed set of three numbers in positions 1 to 3 is the same, then we can extend our argument to say that exactly $\frac{1}{3}$ of all arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have a local peak across positions 1 to 3.
This argument applies to any of the 6 possible places in which a local peak can occur.
Therefore, the average number of local peaks in all of the arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is $6 \times \frac{1}{3} = 2$.