Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2005 Fryer Contest

Wednesday, April 20, 2005

Solutions

1. (a) The area of a circle with radius r is πr^2.
So the area of the larger circle is $\pi(10^2) = 100\pi$ and the area of the smaller circle is $\pi(6^2) = 36\pi$.
The area of the ring between the two circles is the difference of these two areas.
Therefore, the area of the ring is $100\pi - 36\pi = 64\pi$.

(b) The area of the inside circle (region X) is $\pi(4^2) = 16\pi$.
Using a similar technique to part (a), the area of the middle ring (region Y) is
$\pi(6^2) - \pi(4^2) = 36\pi - 16\pi = 20\pi$.
Also, the area of the outer ring (region Z) is $\pi(7^2) - \pi(6^2) = 49\pi - 36\pi = 13\pi$.
Therefore, region Y has the largest area.

(c) The area of the ring between the two largest circles is $\pi(13^2) - \pi(12^2) = 169\pi - 144\pi = 25\pi$.
Suppose that the radius of the smallest circle is r.
Thus, the area of the smallest circle is πr^2.
Since the area of the smallest circle is equal to the area of the ring between the two largest circles, then $\pi r^2 = 25\pi$ so $r^2 = 25$ and so $r = 5$ since $r > 0$.
Therefore, the radius of the smallest circle is 5.

2. (a) Anh goes first and puts an A in the middle box.
According to the rules, Bharati must put her initial in one or two boxes which are next to each other. So Bharati can only put a B in one of the two empty boxes, since they are not next to each other.

B	A	

	A	B

This leaves one empty box, and Anh wins by putting an A in this box.
Therefore, Anh will win no matter what Bharati does.

(b) *Solution 1*
Suppose Anh puts an A in the rightmost box.

B		A		A

Then Bharati can only put a B in one of the two empty boxes, since they are not next to each other.
This leaves one empty box, and Anh will win by putting an A in this empty box.
Therefore, Anh is guaranteed to win if he puts an A in the rightmost box.

Solution 2
Suppose Anh puts an A in the second box from the right end.

B		A	A	

Then Bharati can only put a B in one of the two empty boxes, since they are not next to each other.
This leaves one empty box, and Anh will win by putting an A in this empty box.
Therefore, Anh is guaranteed to win if he puts an A in the second box from the right.

Solution 3
Suppose Anh puts an A in the rightmost box.

B		A		A

We can now remove the leftmost and rightmost boxes, leaving us with three boxes with an A in the middle and it being Bharati's turn.
This leaves us in the situation of part (a), so Anh will be guaranteed to win.

(c) Possibility #1
Suppose Bharati puts a B in the two rightmost boxes.

	A		B	B

Then Anh can only put an A in one of the two empty boxes, since they are not next to each other.
This leaves one empty box, and Bharati wins by putting a B in this empty box.
Therefore, Bharati is guaranteed to win in this case.

Possibility #2
Suppose Bharati puts a B in the middle box and the box to its right.

	A	B	B	

Then Anh can only put an A in one of the two empty boxes, since they are not next to each other.
This leaves one empty box, and Bharati wins by putting a B in this empty box.
Therefore, Bharati is guaranteed to win in this case.

These are the two possible moves that Bharati can make next to guarantee she wins.
(The only other four possible moves are putting a B in any one of the empty boxes. Why do each of these four moves allow A to win?)

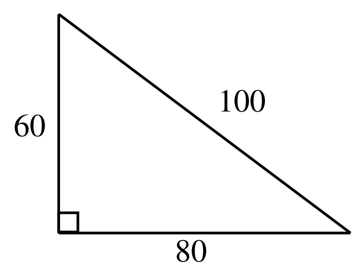
3. (a) *Solution 1*
Since the side lengths of a Nakamoto triangle are in the ratio $3:4:5$, then the side lengths must be the products 3, 4 and 5 with the same integer.
For one of the sides to have a length of 28, it must be the multiple of 4, since 28 is not a multiple of 3 or 5.
Since $28 = 4 \times 7$, then the three side lengths must be $3 \times 7 = 21$, 28 and $5 \times 7 = 35$.

Solution 2
Since the side lengths of a Nakamoto triangle are three integers in the ratio $3:4:5$, then the side lengths are $3x$, $4x$ and $5x$, for some positive integer x.
Since one of these three sides is 28, then we must have $4x = 28$ (or $x = 7$), because 28 is not a multiple of 3 or 5.
Therefore, the side lengths are $3 \times 7 = 21$, 28 and $5 \times 7 = 35$.

(b) *Solution 1*
Since the side lengths of a Nakamoto triangle are in the ratio $3:4:5$, then the side lengths must be the products 3, 4 and 5 with the same integer.
The Nakamoto triangle with the shortest sides is that with side lengths 3, 4 and 5, which has a perimeter of $3+4+5=12$.
Since we are given the Nakamoto triangle of perimeter 96 and $96=12\times 8$, then we must multiply each of the side lengths of the triangle with sides 3, 4 and 5 by 8 to get a perimeter of 96.
Therefore, the side lengths are $3\times 8=24$, $4\times 8=32$ and $5\times 8=40$.

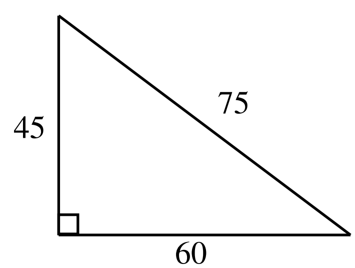
Solution 2
Since the side lengths of a Nakamoto triangle are three integers in the ratio $3:4:5$, then the side lengths are $3x$, $4x$ and $5x$, for some integer x.
Since the perimeter is 96, then $3x+4x+5x=96$ or $12x=96$ or $x=8$.
Therefore, the side lengths are $3\times 8=24$, $4\times 8=32$ and $5\times 8=40$.

(c) Since 60 is divisible by 3, then there is a Nakamoto triangle with a side length of 60 in the "3" position. Since $60=3\times 20$, then the side lengths of this triangle are 60, $4\times 20=80$ and $5\times 20=100$.
Since the ratio of the side lengths are $3:4:5$, then this triangle is right-angled. In fact, since $60^2+80^2=100^2$, then by the Pythagorean Theorem, the right angle is between the sides of lengths 60 and 80. (Alternatively, we could have said that since the triangle is right-angled and 100 is the longest side, then 100 must be the hypotenuse, so the right angle is between the sides of lengths 60 and 80.)



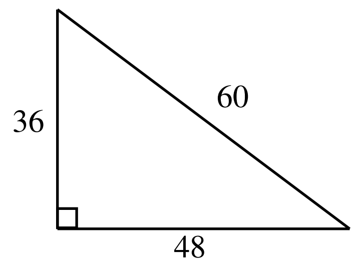
Therefore, the area of this triangle is $\frac{1}{2}bh=\frac{1}{2}(60)(80)=2400$.

Since 60 is divisible by 4, then there is a Nakamoto triangle with a side length of 60 in the "4" position. Since $60=4\times 15$, then the side lengths of this triangle are $3\times 15=45$, 60 and $5\times 15=75$.
Since the triangle is right-angled and 75 is the longest side, then 75 must be the hypotenuse, so the right angle is between the sides of lengths 45 and 60.



Therefore, the area of this triangle is $\frac{1}{2}bh=\frac{1}{2}(45)(60)=1350$.

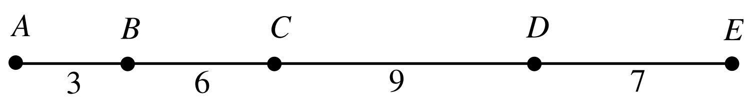
Since 60 is divisible by 5, then there is a Nakamoto triangle with a side length of 60 in the "5" position. Since $60 = 5 \times 12$, then the side lengths of this triangle are $3 \times 12 = 36$, $4 \times 12 = 48$ and 60.
Since the triangle is right-angled and 60 is the longest side, then 60 must be the hypotenuse, so the right angle is between the sides of lengths 36 and 48.



Therefore, the area of this triangle is $\frac{1}{2}bh = \frac{1}{2}(36)(48) = 864$.

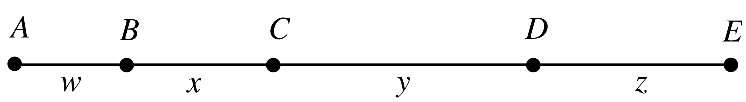
Thus, the possible areas of a Nakamoto triangle with a side length of 60 are 2400, 1350 and 864.

4. (a)



Here, $AB = 3$, $AC = 9$, $AD = 18$, $AE = 25$, $BC = 6$, $BD = 15$, $BE = 22$, $CD = 9$, $CE = 16$, $DE = 7$.
Therefore, the super-sum of AE is $3 + 9 + 18 + 25 + 6 + 15 + 22 + 9 + 16 + 7 = 130$.

(b) *Solution 1*
If the sub-segments have lengths 1 to 10, then the super-sum is
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$.
Suppose that the basic sub-segments have lengths $AB = w$, $BC = x$, $CD = y$ and $DE = z$.



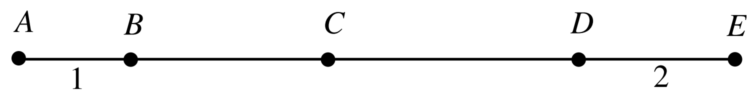
Since all of the sub-segments have integer lengths, then all of the basic sub-segments must have integer length.
What are the lengths of the sub-segments in terms of w, x, y and z?
We have $AB = w$, $AC = w + x$, $AD = w + x + y$, $AE = w + x + y + z$, $BC = x$, $BD = x + y$, $BE = x + y + z$, $CD = y$, $CE = y + z$, and $DE = z$.
Then the super-sum of AE is

$$w+(w+x)+(w+x+y)+(w+x+y+z)+x+(x+y)+(x+y+z)+y+(y+z)+z = 4w+6x+6y+4z$$

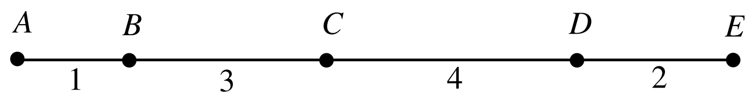
Therefore, the super-sum is always an even number since
$4w + 6x + 6y + 4z = 2(2w + 3x + 3y + 2z)$.
This tells us that the super-sum cannot be 55.
Thus, it is impossible for the line segment AE to have sub-segments of lengths 1 to 10.

Solution 2
Since the longest sub-segment of AE is AE, then we must have $AE = 10$.

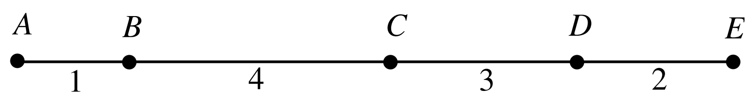
Since the sum of the lengths of the basic sub-segments is the length of AE and the basic sub-segments are included in the list of sub-segments, then the lengths of the sub-segments must be 1, 2, 3, and 4.
In order to get a sub-segment of length 9, we must have the 2, 3 and 4 adjacent, so the 1 must be on one end.
In order to get a sub-segment of length 8, we must have the 1, 3 and 4 adjacent, so the 2 must be on the other end.



This leaves us with two possibilities:

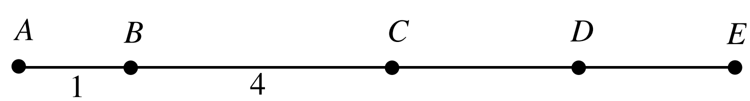


and



In the first possibility, there is no sub-segment of length 5 and two sub-segments of length 4 (AC and CD).
In the second possibility, there is no sub-segment of length 6 and two sub-segments of length 5 (AC and CE).
Having checked all possibilities, we see that it is impossible for the line segment AE to have sub-segments of lengths 1 to 10.

Solution 3
Since the longest sub-segment of AE is AE, then we must have $AE = 10$.
Since the sum of the lengths of the basic sub-segments is the length of AE and the basic sub-segments are included in the list of sub-segments, then the lengths of the sub-segments must be 1, 2, 3, and 4.
In order to avoid repeating lengths of sub-segments, we cannot have the 2 or 3 next to the 1.
Therefore, we must have the 1 on one end with the 4 next to it.



But this means that the 2 and the 3 are next to each other, so we have two sub-segments of length 5, which is impossible.
Therefore, it is impossible for the line segment AE to have sub-segments of lengths 1 to 10.

(c) *Solution 1*
What happens to the super-sum when we first add a basic sub-segment JK of length $\frac{1}{10}$?
The sub-segments of AK include all of the sub-segments of AJ (which have a combined length of 45), and then some additional sub-segments.
The additional sub-segments are ones which are sub-segments of AK but not of AJ. In other words, they are sub-segments which include the basic sub-segment JK. These are the sub-segments JK, IK, HK, and so on, all the way to BK and AK.

What are the lengths of these additional sub-segments?
$JK = \frac{1}{10}$
$IK = IJ + JK = \frac{1}{9} + \frac{1}{10}$
$HK = HI + IJ + JK = \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$
$\vdots$
$AK = AB + BC + \cdots + IJ + JK = 1 + \frac{1}{2} + \cdots + \frac{1}{9} + \frac{1}{10}$
All 10 of these additional sub-segments contain the basic sub-segment JK, 9 contain IJ, 8 contain HI, and so on, with 2 containing BC and 1 containing AB.
Therefore, the super-sum of AK is

$$45 + 10\left(\frac{1}{10}\right) + 9\left(\frac{1}{9}\right) + \cdots + 2\left(\frac{1}{2}\right) + 1(1) = 45 + 10 = 55$$

What happens when we add a basic sub-segment KL of length $\frac{1}{11}$?
As above, this will add to the sub-segments already contained in AK an additional 11 sub-segments containing KL, 10 containing JK, and so on, with 1 containing AB.
Thus, the super-sum of AL will be

$$55 + 11\left(\frac{1}{11}\right) + 10\left(\frac{1}{10}\right) + \cdots + 1(1) = 55 + 11 = 66$$

Similarly, as we add the final four basic sub-segments to obtain AP, we will add 12, then 13, then 14, then 15 to the super-sum.
Therefore, the super-sum of AP is $66 + 12 + 13 + 14 + 15 = 120$.

Solution 2
Each sub-segment of AP is made up of a number of neighbouring basic sub-segments. We can determine the super-sum of AP by counting the number of sub-segments in which each basic sub-segment occurs, and so determine the contribution of each basic sub-segment to the super-sum.
The basic sub-segment AB occurs in sub-segments AB, AC, …, AP, or 15 sub-segments in total.
By symmetry, OP will also occur in 15 sub-segments.
The basic sub-segment BC occurs in sub-segments BC, BD, …, BP (14 in total) and AC, AD, …, AP (another 14 in total), for a total of 28 sub-segments.
By symmetry, NO will also occur in 28 sub-segments.
Is there a better way to count these total number of sub-segments which contain a given basic sub-segment without having to list them all?
Consider BC again.
A sub-segment containing BC must have left endpoint A or B (2 possibilities) and right endpoint from C to P (14 possibilities). Each combination of left endpoint and right endpoint is possible, so there are $2 \times 14 = 28$ possible sub-segments. (The same argument applies for NO with 14 possible left endpoints and 2 possible right endpoints.)
We make a table containing each of the remaining basic sub-segments, the number of possible left endpoints for a sub-segment containing it, the number of possible right endpoints, and the total number of sub-segments:

Basic sub-segment	# Possible L endpoints	# Possible R endpoints	Total # of sub-segments
CD	3	13	39
DE	4	12	48
EF	5	11	55
FG	6	10	60
GH	7	9	63
HI	8	8	64
IJ	9	7	63
JK	10	6	60
KL	11	5	55
LM	12	4	48
MN	13	3	39

Therefore, the super-sum is
$15(1) + 28(\frac{1}{2}) + 39(\frac{1}{3}) + 48(\frac{1}{4}) + 55(\frac{1}{5}) + 60(\frac{1}{6}) + 63(\frac{1}{7}) + 64(\frac{1}{8}) + 63(\frac{1}{9}) + 60(\frac{1}{10}) + 55(\frac{1}{11}) + 48(\frac{1}{12}) + 39(\frac{1}{13}) + 28(\frac{1}{14}) + 15(\frac{1}{15})$
or

$$15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Therefore, the super-sum of AP is 120.