Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2004 Solutions
Euclid Contest

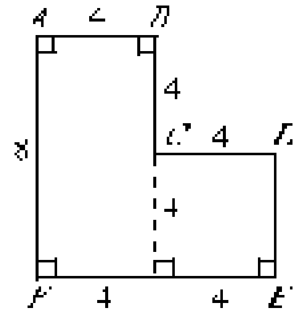
for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. (a) *Solution*

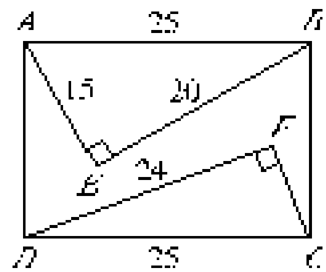
Because all of the angles in the figure are right angles, then $BC = DE = 4$.
Thus, we can break up the figure into a 4 by 8 rectangle and a 4 by 4 square, by extending BC to hit FE. Therefore, the area of the figure is $(8)(4)+(4)(4)=48$.



Answer: 48

(b) *Solution*

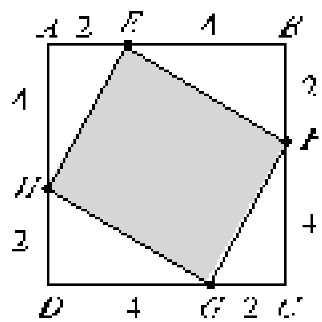
By the Pythagorean Theorem in triangle ABE, $AB^2 = 15^2 + 20^2 = 625$, so $AB = 25$.
Since $ABCD$ is a rectangle, $CD = AB = 25$, so by the Pythagorean Theorem in triangle CFD, we have $625 = 25^2 = 24^2 + CF^2$, so $CF^2 = 625 - 576 = 49$, or $CF = 7$.



Answer: 7

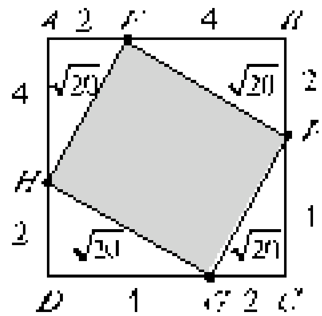
(c) *Solution 1*

Since $ABCD$ is a square of side length 6 and each of $AE:EB$, $BF:FC$, $CG:GD$, and $DH:HA$ is equal to 1 : 2, then $AE = BF = CG = DH = 2$ and $EB = FC = GD = HA = 4$.
Thus, each of the triangles HAE, EBF, FCG, and GDH is right-angled, with one leg of length 2 and the other of length 4.
Then the area of $EFGH$ is equal to the area of square $ABCD$ minus the combined area of the four triangles, or $6^2 - 4\left[\frac{1}{2}(2)(4)\right] = 36 - 16 = 20$ square units.



Solution 2

Since $ABCD$ is a square of side length 6 and each of $AE : EB$, $BF : FC$, $CG : GD$, and $DH : HA$ is equal to 1 : 2, then $AE = BF = CG = DH = 2$ and $EB = FC = GD = HA = 4$.



Thus, each of the triangles HAE, EBF, FCG, and GDH is right-angled, with one leg of length 2 and the other of length 4.

By the Pythagorean Theorem,

$EF = FG = GH = HE = \sqrt{2^2 + 4^2} = \sqrt{20}$.

Since the two triangles HAE and EBF are congruent (we know the lengths of all three sides of each), then $\angle AHE = \angle BEF$. But $\angle AHE + \angle AEH = 90^\circ$, so $\angle BEF + \angle AEH = 90^\circ$, so $\angle HEF = 90^\circ$.

In a similar way, we can show that each of the four angles of $EFGH$ is a right-angle, and so $EFGH$ is a square of side length $\sqrt{20}$.

Therefore, the area of $EFGH$ is $\left(\sqrt{20}\right)^2 = 20$ square units.

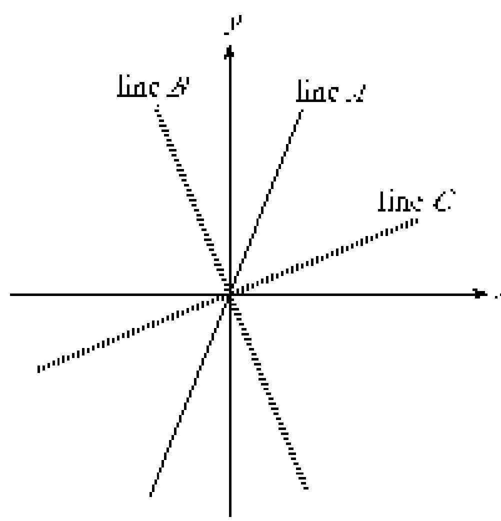
2. (a) Rearranging the equation of the given line $3x - y = 6$, we get $y = 3x - 6$, so the given line has y-intercept –6.

Since horizontal lines have the general form $y = a$ for some constant a, then the horizontal line with y-intercept –6 is the line $y = -6$.

Answer: $y = -6$

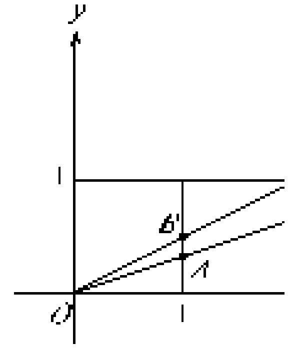
(b) When line A with equation $y = 2x$ is reflected in the y-axis, the resulting line (line B) has equation $y = -2x$. (Reflecting a line in the y-axis changes the sign of the slope.)

Since the slope of line B is –2 and line C is perpendicular to line B, then the slope of line C is $\frac{1}{2}$ (the slopes of perpendicular lines are negative reciprocals).



Answer: $\frac{1}{2}$

(c) *Solution 1*



Consider the line through O and P. To get from O to P, we go right 2 and up 1. Since B lies on this line and to get from O to B we go over 1, then we must go up $\frac{1}{2}$, to keep the ratio constant.

Consider the line through O and Q. To get from O to Q, we go right 3 and up 1. Since A lies on this line and to get from O to A we go over 1, then we must go up $\frac{1}{3}$, to keep the ratio constant.

Therefore, since A and B lie on the same vertical line, then $AB = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Solution 2

Since the line through P passes through the origin, then its equation is of the form $y = mx$. Since it passes through the point $(2,1)$, then $1 = 2m$, so the line has equation $y = \frac{1}{2}x$. Since B has x-coordinate 1, then $y = \frac{1}{2}(1) = \frac{1}{2}$, so B has coordinates $\left(1,\frac{1}{2}\right)$.

Similarly, we can determine that the equation of the line through Q is $y = \frac{1}{3}x$, and so A has coordinates $\left(1,\frac{1}{3}\right)$.

Therefore, since A and B lie on the same vertical line, then $AB = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

3. (a) *Solution 1*

If the sequence has common difference d, then we can write the sequence as $a-2d$, $a-d$, a, $a+d$, $a+2d$.

From the given information, $(a-2d)+(a-d) = 2$, or $2a-3d = 2$.

Also, $(a+d)+(a+2d) = -18$, or $2a+3d = -18$.

We want to determine the third term, which is a.

We can get an equation involving a only by adding the two equations to get $4a = (2a-3d)+(2a+3d) = 2+(-18) = -16$, and so a, the third term, is -4.

Solution 2

If the sequence has first term a and common difference d, then we can write the sequence as a, $a+d$, $a+2d$, $a+3d$, $a+4d$.

From the given information, $a+(a+d) = 2$, or $2a+d = 2$.

Also, $(a+3d)+(a+4d) = -18$, or $2a+7d = -18$.

Subtracting these two equations, we obtain $6d = -20$ or $d = -\frac{10}{3}$.

Substituting back into the first equation, $2a = 2-\left(-\frac{10}{3}\right) = \frac{16}{3}$ and so $a = \frac{8}{3}$.

Therefore, the third term is $a+2d = \frac{8}{3}+2\left(-\frac{10}{3}\right) = -\frac{12}{3} = -4$.

Answer: -4

(b) *Solution 1*

Since $(x-y)^2 = x^2 - 2xy + y^2$ and $(x+y)^2 = x^2 + 2xy + y^2$, then $(x+y)^2 - (x-y)^2 = 4xy$.

Thus, we have $(x+y)^2 = \left(4\sqrt{2}\right)^2 + 4(56) = 32 + 224 = 256$, and so $x + y = 16$ or $x + y = -16$.

Since we are told that there are two values, then these two values must be 16 and –16.

Solution 2

From the first equation, $x = y + 4\sqrt{2}$, so substituting into the second equation,

$$\left(y + 4\sqrt{2}\right)y = 56$$

$$y^2 + 4\sqrt{2}y - 56 = 0$$

$$y = \frac{-4\sqrt{2} \pm \sqrt{\left(4\sqrt{2}\right)^2 - 4(1)(-56)}}{2}$$

$$y = \frac{-4\sqrt{2} \pm \sqrt{256}}{2}$$

$$y = -2\sqrt{2} \pm 8$$

If $y = -2\sqrt{2} + 8$, then $x = y + 4\sqrt{2} = 2\sqrt{2} + 8$, so $x + y = 16$.

If $y = -2\sqrt{2} - 8$, then $x = y + 4\sqrt{2} = 2\sqrt{2} - 8$, so $x + y = -16$.

Therefore, the two values are 16 and –16.

4. (a) *Solution 1*

There are 36 possibilities for the pair of numbers on the faces when the dice are thrown.

For the product of the two numbers, each of which is between 1 and 6, to be divisible by 5, one of the two numbers must be equal to 5.

Therefore, the possible pairs for the faces are

$$(1,5),\ (2,5),\ (3,5),\ (4,5),\ (5,5),\ (6,5),\ (5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,6)$$

ie. there are 11 possibilities.

Thus, the probability is $\frac{11}{36}$.

Solution 2

For the product of the two numbers, each of which is between 1 and 6, to be divisible by 5, one of the two numbers must be equal to 5.

When the two dice are thrown, the probability that the first die has a 5 on the top face and any number appears on the second die has any number on the top face is $\frac{1}{6} \times 1 = \frac{1}{6}$.

Also, the probability that any number appears on the first die and a 5 appears on the second die is $1 \times \frac{1}{6} = \frac{1}{6}$.

If we consider the sum of these probabilities, we have double-counted the possibility that a 5 occurs on both dice, which happens with probability $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
Therefore, the required probability is $\frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$.

Answer: $\frac{11}{36}$

(b) First, we compute an expression for the composition of the two given functions:

$$\begin{aligned} f(g(x)) &= f(ax+b) \\ &= (ax+b)^2 - (ax+b) + 2 \\ &= a^2x^2 + 2abx + b^2 - ax - b + 2 \\ &= a^2x^2 + (2ab - a)x + \left(b^2 - b + 2\right) \end{aligned}$$

But we already know that $f(g(x)) = 9x^2 - 3x + 2$, so comparing coefficients, we see that

$$\begin{aligned} a^2 &= 9 \qquad (1) \\ 2ab - a &= -3 \qquad (2) \\ b^2 - b + 2 &= 2 \qquad (3) \end{aligned}$$

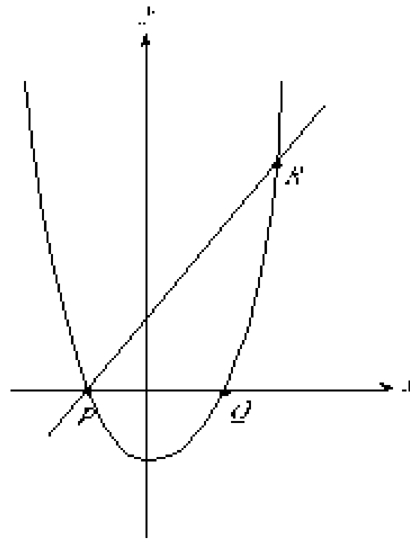
From the first equation, $a = 3$ or $a = -3$.
From the third equation, $b^2 - b = b(b-1) = 0$ so $b = 0$ or $b = 1$.
There are thus 4 possible pairs (a, b) which could solve the problem. We will check which pairs work by looking at the second equation.
From the second equation, $a(2b-1) = -3$, so if $a = 3$ then $b = 0$, and if $a = -3$ then $b = 1$.
Therefore, the possible ordered pairs (a, b) are $(3,0)$ and $(-3,1)$.

5. (a)

$$\begin{aligned} 16^x &= 2^{x+5} - 2^{x+4} \\ \left(2^4\right)^x &= 2^{x+4}\left(2^1 - 1\right) \\ 2^{4x} &= 2^{x+4}(1) \\ 2^{4x} &= 2^{x+4} \\ 4x &= x + 4 \qquad \text{(equating exponents)} \\ 3x &= 4 \\ x &= \tfrac{4}{3} \end{aligned}$$

Answer: $x = \frac{4}{3}$

(b) Point P is the point where the line $y = 3x + 3$ crosses the $x$axis, and so has coordinates $(-1,0)$.



Therefore, one of the roots of the parabola $y = x^2 + tx - 2$ is $x = -1$, so

$$0 = (-1)^2 + t(-1) - 2$$
$$0 = 1 - t - 2$$
$$t = -1$$

The parabola now has equation $y = x^2 - x - 2 = (x+1)(x-2)$ (we already knew one of the roots so this helped with the factoring) and so its two x-intercepts are –1 and 2, ie. P has coordinates $(-1,0)$ and Q has coordinates $(2,0)$.

We now have to find the coordinates of the point R. We know that R is one of the two points of intersection of the line and the parabola, so we equate their equations:

$$3x + 3 = x^2 - x - 2$$
$$0 = x^2 - 4x - 5$$
$$0 = (x+1)(x-5)$$

(Again, we already knew one of the solutions to this equation ($x = -1$) so this made factoring easier.) Since R does not have x-coordinate –1, then R has x-coordinate $x = 5$. Since R lies on the line, then $y = 3(5) + 3 = 18$, so R has coordinates $(5,18)$.

We can now calculate the area of triangle PQR. This triangle has base of length 3 (from P to Q) and height of length 18 (from the x-axis to R), and so has area $\frac{1}{2}(3)(18) = 27$.

Thus, $t = -1$ and the area of triangle PQR is 27.

6. (a) In order to use as many coins as possible, Lori should use coins with smaller values wherever possible.

Can Lori make \$1.34 without using the loonie? The total value of all of the other coins is $3(\$0.25) + 3(\$0.10) + 3(\$0.05) + 5(\$0.01) = \$1.25$, so Lori needs to use the loonie to pay for the toy helicopter. Thus, she needs to try to make \$0.34 with as many coins as possible.

Next, Lori must use 4 pennies in order to make \$0.34 (since each other coin's value is a multiple of 5 cents). So she has used 5 coins thus far, and still needs to try to make \$0.30 with as many coins as possible.

In order to make \$0.30 using as many of the quarters, dimes and nickels as possible, she should use 2 nickels and 2 dimes, or 4 coins. (If a quarter is used, a single nickel completes the \$0.30 with 2 coins only. At least 2 dimes must be used.)

Therefore, the maximum number of coins she can use is 9.

Answer: 9

(b) When the dimensions were increased by n % from 10 by 15, the new dimensions were $10\left(1+\frac{n}{100}\right)$ by $15\left(1+\frac{n}{100}\right)$.

When the resolution was decreased by n percent, the new resolution was $75\left(1-\frac{n}{100}\right)$.

(Note that n cannot be larger than 100, since the resolution cannot be decreased by more than 100%.)

Therefore, the number of pixels in the new image is

$$\left[10\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right]\times\left[15\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right]$$

Since we know that the number of pixels in the new image is 345 600, then

$$\begin{aligned}
\left[10\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right]\times\left[15\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right] &= 345600 \\
[10\times 75]\times[15\times 75]\times\left(1+\frac{n}{100}\right)^2\times\left(1-\frac{n}{100}\right)^2 &= 345600 \\
843750\left(1+\frac{n}{100}\right)^2\left(1-\frac{n}{100}\right)^2 &= 345600 \\
\left(1-\frac{n^2}{100^2}\right)^2 &= 0.4096 \\
1-\frac{n^2}{100^2} &= \pm 0.64 \\
1-\frac{n^2}{100^2} &= 0.64 && (n \text{ cannot be larger than } 100) \\
\frac{n^2}{100^2} &= 0.36 \\
\frac{n}{100} &= 0.6 && (\text{since } n \text{ must be positive}) \\
n &= 60
\end{aligned}$$

Thus, $n = 60$.

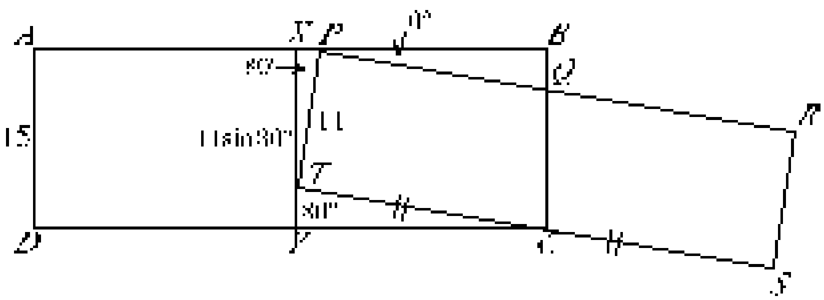
7. (a) We first calculate the length of AC using the cosine law:

$$\begin{aligned}
AC^2 &= 7^2 + 8^2 - 2(7)(8)\cos(120^\circ) \\
AC^2 &= 49 + 64 - 112\left(-\tfrac{1}{2}\right) \\
AC^2 &= 169 \\
AC &= 13
\end{aligned}$$

Since triangle ABC is right-angled and isosceles, then $x = AB = \sqrt{2}(AC) = 13\sqrt{2}$.

Answer: $x = 13\sqrt{2}$

(b) First, we draw a line through T which is perpendicular to AB. This line cuts AB at X and CD at Y.
Since $\angle TPR$ is a right angle, then $\angle XPT = 80^\circ$. Thus, $XT = 11\sin(80^\circ)$.
Since $XY = 15$, then $TY = 15 - 11\sin(80^\circ)$.



But triangle XPT is right-angled at X, so since $\angle XPT = 80^\circ$, then $\angle XTP = 10^\circ$, and so $\angle YTC = 80^\circ$, since $\angle TYC = 90^\circ$.

Thus, $TC = \dfrac{TY}{\cos(80^\circ)} = \dfrac{15 - 11\sin(80^\circ)}{\cos(80^\circ)}$, and so the length of the drawer is

$$TS = 2TC = \frac{30 - 22\sin(80^\circ)}{\cos(80^\circ)} \approx 47.9949.$$

Thus, to the nearest tenth of a centimetre, the length of the drawer is 48.0 cm.
[Note that there are many different ways to do this problem.]

8. (a) Consider the right side of the given equation:

$$\begin{aligned} T^3 + bT + c &= \left(x^2 + \frac{1}{x^2}\right)^3 + b\left(x^2 + \frac{1}{x^2}\right) + c \\ &= \left(x^4 + 2 + \frac{1}{x^4}\right)\left(x^2 + \frac{1}{x^2}\right) + b\left(x^2 + \frac{1}{x^2}\right) + c \\ &= x^6 + 3x^2 + \frac{3}{x^2} + \frac{1}{x^6} + b\left(x^2 + \frac{1}{x^2}\right) + c \\ &= x^6 + \frac{1}{x^6} + (b+3)\left(x^2 + \frac{1}{x^2}\right) + c \end{aligned}$$

For this expression to be equal to $x^6 + \dfrac{1}{x^6}$ for all values of x, we want $b + 3 = 0$ or $b = -3$ and $c = 0$.

(b) *Solution 1*

We start with $x^3 + \frac{1}{x^3} = 2\sqrt{5}$ and square both sides:

$$\left(x^3 + \frac{1}{x^3}\right)^2 = \left(2\sqrt{5}\right)^2$$

$$x^6 + 2 + \frac{1}{x^6} = 20$$

$$x^6 + \frac{1}{x^6} = 18$$

Using the result from part (a) and letting $T = x^2 + \frac{1}{x^2}$, we see that $T^3 - 3T = 18$.

So we would like to factor the equation $T^3 - 3T - 18 = 0$.

After some trial and error, we can see that $T = 3$ is a solution, so by the Factor Theorem, $(T-3)\left(T^2 + 3T + 6\right) = 0$, and $T^2 + 3T + 6$ has no real roots.

Therefore, $T = 3$, ie. $x^2 + \frac{1}{x^2} = 3$.

Solution 2

Let $t = x + \frac{1}{x}$. Since we saw in (a) that $x^6 + \frac{1}{x^6} = \left(x^2 + \frac{1}{x^2}\right)^3 - 3\left(x^2 + \frac{1}{x^2}\right)$, then it makes sense that $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = t^3 - 3t$.

Therefore, we have $t^3 - 3t = 2\sqrt{5}$ or $t^3 - 3t - 2\sqrt{5} = 0$.

Since $\left(\sqrt{5}\right)^3 = 5\sqrt{5}$, then $t = \sqrt{5}$ is a solution to this equation, so factoring we obtain $\left(t - \sqrt{5}\right)\left(t^2 + \sqrt{5}t + 2\right) = 0$. The quadratic factor has discriminant $\left(\sqrt{5}\right)^2 - 4(1)(2) = -3 < 0$ and so has no real roots.

Therefore, $t = x + \frac{1}{x} = \sqrt{5}$.

Squaring, we obtain

$$\left(x + \frac{1}{x}\right)^2 = \left(\sqrt{5}\right)^2$$

$$x^2 + 2 + \frac{1}{x^2} = 5$$

$$x^2 + \frac{1}{x^2} = 3$$

Solution 3

We start with $x^3+\frac{1}{x^3}=2\sqrt{5}$ and square both sides:

$$\left(x^3+\frac{1}{x^3}\right)^2=\left(2\sqrt{5}\right)^2$$
$$x^6+2+\frac{1}{x^6}=20$$
$$x^6+\frac{1}{x^6}=18$$

From part (a), if $T=x^2+\frac{1}{x^2}$, then $T^3=\left(x^2+\frac{1}{x^2}\right)^3=x^6+3\left(x^2+\frac{1}{x^2}\right)+\frac{1}{x^6}=18+3T$.

So we would like to factor the equation $T^3-3T-18=0$.

After some trial and error, we can see that $T=3$ is a solution, so by the Factor Theorem, $(T-3)\left(T^2+3T+6\right)=0$, and T^2+3T+6 has no real roots.

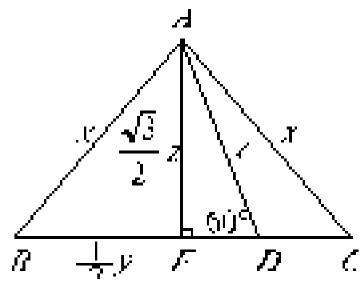
Therefore, $T=3$, ie. $x^2+\frac{1}{x^2}=3$.

9. (a) *Solution 1*

From A, drop a perpendicular to BC.

From triangle ABE, $AE^2=x^2-\frac{1}{4}y^2$.

From triangle ADE, $AE=\frac{\sqrt{3}}{2}z$ or $AE^2=\frac{3}{4}z^2$.



Equating these two expressions, we get $x^2-\frac{1}{4}y^2=\frac{3}{4}z^2$ or $4x^2-y^2=3z^2$.

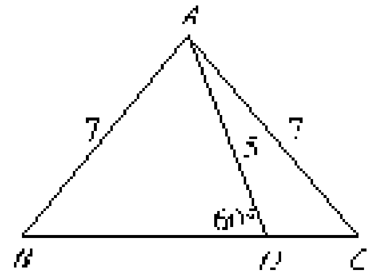
If $x=7$ and $z=5$, then

$$196-y^2=75$$
$$y^2=121$$
$$y=\pm 11$$

Therefore, the Kirk triplet with $x=7$ and $z=5$ is $(7,11,5)$.

Solution 2

By the cosine law in triangle *ADB*,



$$7^2 = 5^2 + BD^2 - 2(5)(BD)\cos(60^\circ)$$
$$0 = BD^2 - 5BD - 24$$
$$0 = (BD - 8)(BD + 3)$$

Since *BD* is a length, $BD = 8$.

Since $\angle ADC = 120^\circ$, then by the cosine law in triangle *ADC*,

$$7^2 = 5^2 + DC^2 - 2(5)(DC)\cos(120^\circ)$$
$$0 = DC^2 + 5DC - 24$$
$$0 = (DC + 8)(DC - 3)$$

Since *DC* is a length, then $DC = 3$. Thus, $y = BD + DC = 11$.

Therefore, the Kirk triplet with $x = 7$ and $z = 5$ is $(7,11,5)$.

(b) *Solution 1*

If $4x^2 - y^2 = 3z^2$ (from (a) Solution 1) and $z = 5$, then

$$4x^2 - y^2 = 75$$
$$(2x + y)(2x - y) = 75$$

Each of $2x + y$ and $2x - y$ is a positive integer and their product is 75. Note that $2x + y$ is bigger than $2x - y$. The factors of 75 are 1, 3, 5, 15, 25, 75.

Looking at each of the possibilities,

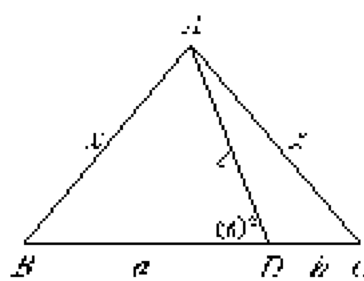
$2x + y$	$2x - y$	x	y
75	1	19	37
25	3	7	11
15	5	5	5

so the two possible Kirk triplets with $z = 5$ are $(19,37,5)$ and $(7,11,5)$.

Solution 2

Let $BD = a$ and $DC = b$.

By the cosine law in triangles *ABD* and *ADC*,



$$x^2 = 5^2 + a^2 - 2(5)(a)\cos(60^\circ) \quad \text{and}$$
$$x^2 = a^2 - 5a + 25$$
$$x^2 = 5^2 + b^2 - 2(5)(b)\cos(120^\circ)$$

$$x^2 = 5^2 + b^2 - 2(5)(b)\cos(120^\circ)$$

$$x^2 = b^2 + 5b + 25$$

Subtracting the second equation from the first, we obtain

$$0 = a^2 - b^2 - 5a - 5b$$

$$0 = (a+b)(a-b-5)$$

$$0 = y(a-b-5)$$

Since y is not 0, then $a = b+5$.

We know $y = a+b = 2b+5$, so $2b = y-5$.

But $x^2 = b^2 + 5b + 25$, so

$$4x^2 = 4b^2 + 20b + 100$$

$$4x^2 = (y-5)^2 + 10(y-5) + 100$$

$$4x^2 = y^2 + 75$$

$$4x^2 - y^2 = 75$$

and thus we continue as in Solution 1 to determine that the only Kirk triplets with $z = 5$ are $(19,37,5)$ and $(7,11,5)$.

(c) In order determine the appropriate Kirk triplet, we need to find a way to determine Kirk triplets. We model our approach after that in part (b), Solution 1.

Drop a perpendicular from A to F on BC.

Since triangle ABC is isosceles, then F is the midpoint of BC.

Also, triangle ADF is a 30-60-90 triangle, with $AD = z$.

Thus, $FD = \frac{1}{2}z$ and $AF = \frac{\sqrt{3}}{2}z$.

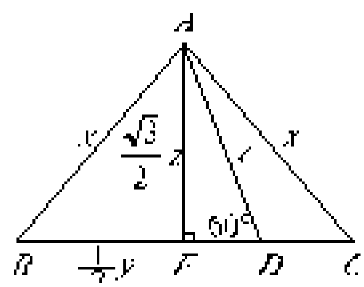
Then triangle ABF is right-angled at F with $AF = \frac{\sqrt{3}}{2}z$, $AB = x$ and $BF = \frac{1}{2}y$.

Thus, by the Pythagorean Theorem,

$$x^2 = \left(\tfrac{1}{2}y\right)^2 + \left(\tfrac{\sqrt{3}}{2}z\right)^2$$

$$4x^2 - y^2 = 3z^2$$

$$(2x+y)(2x-y) = 3z^2$$



Each of $2x+y$ and $2x-y$ is a positive integer and their product is $3z^2$. Note that $2x+y$ is bigger than $2x-y$. Since z is a prime number, the factors of $3z^2$ are 1, 3, z, $3z$, z^2, $3z^2$. (Notice that if z is equal to 2, these are not in ascending order, and if z equals 3, then there is duplication in this list.)

Looking at each of the possibilities,

$2x+y$	$2x-y$	x	y

$3z^2$	1	$\frac{3z^2+1}{4}$	$\frac{3z^2-1}{2}$
z^2	3	$\frac{z^2+3}{4}$	$\frac{z^2-3}{2}$
$3z$	z	z	z

so the only two possible Kirk triplets with a fixed value for z are $\left(\frac{3z^2+1}{4}, \frac{3z^2-1}{2}, z\right)$ and $\left(\frac{z^2+3}{4}, \frac{z^2-3}{2}, z\right)$.

We would like to determine the Kirk triplets with $\cos(\angle ABC)$ is as close to 0.99 as possible.

Looking back at triangle ABF, we see that $\cos(\angle ABC) = \frac{\frac{1}{2}y}{x}$, so in our two cases

$\cos(\angle ABC) = \frac{3z^2-1}{3z^2+1} = 1 - \frac{2}{3z^2+1}$ or $\cos(\angle ABC) = \frac{z^2-3}{z^2+3} = 1 - \frac{6}{z^2+3}$.

In order to make $\cos(\angle ABC)$ close to 0.99, we thus make either $\frac{2}{3z^2+1}$ or $\frac{6}{z^2+3}$ close to 0.01.

In other words we make $3z^2+1$ close to 200, or z^2+3 close to 600.

In the first case, since $3(8)^2+1 = 193$ and $3(9)^2+1 = 244$, and since z must be a prime number, then we try $z = 7$ and $z = 11$, and obtain $\cos(\angle ABC) = \frac{3z^2-1}{3z^2+1}$ to be 0.986487 and 0.994506.

In the second case, since $24^2+3 = 579$ and $25^2+3 = 628$, and since z must be a prime number, then we try $z = 23$ and $z = 29$, and obtain $\cos(\angle ABC) = \frac{z^2-3}{z^2+3}$ to be 0.988722 and 0.993644.

So $\cos(\angle ABC)$ appears to be as close to 0.99 as possible when $z = 23$.

We should double-check to confirm that we actually get a triplet of integers in this case!

In the second case, when $z = 23$, we get the triplet $(133, 263, 23)$.

Thus, the Kirk triplet $(133, 263, 23)$ gives $\cos(\angle ABC)$ is as close to 0.99 as possible.

(Note: We could have proceeded analogously to the second approach to 9(b) to obtain $4x^2 - y^2 = 3z^2$ in this way, and then proceeded as above in this solution.)

10. (a) We start by placing the two 4's. We systematically try each pair of possible positions – from positions 1 and 5 to positions 4 and 8. For each of these positions, we try placing

the two 3's in each pair of possible positions, and then see if the two 2's and two 1's will fit.
(We can reduce our work by noticing that if a Skolem sequence has the two 4's in positions 1 and 5, then reversing the sequence will give a Skolem sequence with the two 4's in positions 4 and 8. So we only need to consider putting the two 4's in positions 1 and 5, and in positions 2 and 6. The remaining possibilities can be dealt with by reversing.)
Thus, the six possible Skolem sequences of order 4 are:

(4, 2, 3, 2, 4, 3, 1, 1) and its reverse, (1, 1, 3, 4, 2, 3, 2, 4)
(4, 1, 1, 3, 4, 2, 3, 2) and its reverse, (2, 3, 2, 4, 3, 1, 1, 4)
(3, 4, 2, 3, 2, 4, 1, 1) and its reverse, (1, 1, 4, 2, 3, 2, 4, 3)

(b) Since we are trying to create a Skolem sequence of order 9, then there are 18 positions to fill with 10 odd numbers and 8 even numbers.
We are told that $s_{18} = 8$, so we must have $s_{10} = 8$, since the 8's must be 8 positions apart. By condition III, between the two 8's, there can be only one odd integer. But there are 7 positions between the two 8's and only 6 remaining even numbers to place. Thus, all 6 remaining even numbers are placed between the two 8's. The only way in which this is possible is with the two 6's next to the two 8's, then the two 4's, then the two 2's. (The two 8's are 8 positions apart, and the two 6's must be 6 positions apart.)
Thus, the sequence so far is:

(__, __, 1, __, __, __, __, __, __, 8, 6, 4, 2, __, 2, 4, 6, 8)

The numbers that we have left to place are 1, 3, 3, 5, 5, 7, 7, 9, 9, and empty positions are 1, 2, 4, 5, 6, 7, 8, 9, 14.
Since the 9's must be 9 positions apart, they must be placed in positions 5 and 14.
Thus, we have

(__, __, 1, __, 9, __, __, __, __, 8, 6, 4, 2, 9, 2, 4, 6, 8)

The remaining 1 must be placed in position 2 or 4. If it is placed in position 2, then the 7's can only go in positions 1 and 8, giving

(7, 1, 1, __, 9, __, __, 7, __, 8, 6, 4, 2, 9, 2, 4, 6, 8)

But we now cannot place both the two 3's and the two 5's. (The placing of one of these pairs means that the other pair cannot be placed.)
We conclude that the only possibility is that the remaining 1 must be placed in position 4. This gives

(__, __, 1, 1, 9, __, __, __, __, 8, 6, 4, 2, 9, 2, 4, 6, 8)

with 3, 3, 5, 5, 7, 7 left to be placed in positions 1, 2, 6, 7, 8, 9.
Now the two 3's must be placed in positions 6 and 9, so the 7's must be placed in positions 1 and 8, and finally the 5's must be placed in positions 2 and 7.
Therefore, the only Skolem sequence satisfying the given conditions is

(7, 5, 1, 1, 9, 3, 5, 7, 3, 8, 6, 4, 2, 9, 2, 4, 6, 8)

(c) *Solution 1*

Assume that there is a Skolem sequence of order n, where n is of the form $4k+2$ or $4k+3$. (We will deal with both cases together.)
Let P_1 be the position number of the left-most 1 in the Skolem sequence. Then P_1+1 is the position number of the other 1.
Similarly, let $P_2,\ P_3,\ \ldots,\ P_n$ be the position numbers of the left-most 2, 3, …, n, respectively, in the Skolem sequence. Then $P_2+2,\ P_3+3,\ \ldots,\ P_n+n$ are the position numbers of the other occurrences of 2, 3, …, n, respectively.
Since the Skolem sequence exists, then the numbers $P_1,\ P_2,\ P_3,\ \ldots,\ P_n,\ P_1+1,\ P_2+2,\ P_3+3,\ \ldots,\ P_n+n$ are a rearrangement of all of the position numbers, ie. the numbers 1, 2, …, $2n$.
Thus,

$$\begin{aligned} P_1+P_2+\cdots+P_n+(P_1+1)+(P_2+2)+\cdots+(P_n+n) &= 1+2+\cdots+2n \\ 2(P_1+P_2+\cdots+P_n)+(1+2+\cdots+n) &= 1+2+\cdots+2n \\ 2(P_1+P_2+\cdots+P_n)+\frac{n(n+1)}{2} &= \frac{2n(2n+1)}{2} \\ 2(P_1+P_2+\cdots+P_n)+\frac{n(n+1)}{2} &= n(2n+1) \qquad (**) \end{aligned}$$

using the fact that $1+2+\cdots+k=\dfrac{k(k+1)}{2}$.
We now look at the parities of the terms in equation (**) in each of our two cases.
If $n=4k+2$, then $\dfrac{n(n+1)}{2}=\dfrac{(4k+2)(4k+3)}{2}=(2k+1)(4k+3)$ which is the product of two odd numbers, so is odd, and $n(2n+1)=(4k+2)(8k+5)$, which is the product of an even number and an odd number, so is even.
Thus, (**) is Even + Odd = Even, a contradiction.

If $n=4k+3$, then $\dfrac{n(n+1)}{2}=\dfrac{(4k+3)(4k+4)}{2}=(4k+3)(2k+2)$ which is the product of an odd number and an even number, so is even, and $n(2n+1)=(4k+3)(8k+7)$, which is the product of two odd numbers, so is odd.
Thus, (**) is Even + Even = Odd, a contradiction.

Therefore, in either case, a contradiction is reached, so there cannot exist a Skolem sequence of order n, if n is of the form $4k+2$ or $4k+3$, where k is a non-negative integer.

Solution 2

Assume that there is a Skolem sequence of order *n*, where *n* is of the form $4k+2$ or $4k+3$.
Let *l* be an integer between 1 and *n*.
If *l* is even, then the two position numbers in which *l* is placed differ by an even number (namely *l*), and so are either both odd or both even.
If *l* is odd, then the two position numbers in which *l* is placed differ by an odd number (namely *l*), and so one is odd and the other is even.

Case 1: $n = 4k+2$

Between 1 and *n*, there are $2k+1$ even numbers $(2, 4, \ldots, 4k+2)$ and $2k+1$ odd numbers $(1, 3, \ldots, 4k+1)$.
The position numbers in a Skolem sequence of order $n = 4k+2$ are the integers 1, 2, $\ldots, 8k+4$, so there are $4k+2$ even position numbers and $4k+2$ odd position numbers.
Considering the $2k+1$ odd numbers in the sequence gives us $2k+1$ odd position numbers and $2k+1$ even position numbers, ie. an odd number of odd position numbers and an odd number of even position numbers.
Each even number in the sequence will contribute either two odd position numbers or two even position numbers. In other words, all of the even numbers in the sequence contribute an even number of odd position numbers.
Thus, the total number of odd position numbers is odd plus even, which is odd. This is a contradiction, because we know there must be $4k+2$ odd position numbers.

Case 2: $n = 4k+3$

Between 1 and *n*, there are $2k+1$ even numbers $(2, 4, \ldots, 4k+2)$ and $2k+2$ odd numbers $(1, 3, \ldots, 4k+3)$.
The position numbers in a Skolem sequence of order $n = 4k+3$ are the integers $1, 2, \ldots, 8k+6$, so there are $4k+3$ even position numbers and $4k+3$ odd position numbers.
Considering the $2k+2$ odd numbers in the sequence gives us $2k+2$ odd position numbers and $2k+2$ even position numbers, ie. an even number of odd position numbers and an even number of even position numbers.
Each even number in the sequence will contribute either two odd position numbers or two even position numbers. In other words, all of the even numbers in the sequence contribute an even number of odd position numbers.
Thus, the total number of odd position numbers is even plus even, which is even. This is a contradiction, because we know there must be $4k+3$ odd position numbers.

Therefore, in either case, a contradiction is reached, so there cannot exist a Skolem sequence of order *n*, if *n* is of the form $4k+2$ or $4k+3$, where *k* is a non-negative integer.