Canadian Mathematics Competition

An activity of The Centre for Education
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2003 Solutions
Hypatia Contest (Grade 11)

1. (a) Let N be the number of tiles in Quentin's possession.
    Since he has 92 extra tiles after forming an n by n square, then
    $$N = n^2 + 92$$
    Since he is 100 tiles short of forming an $(n+2)$ by $(n+2)$ square, then
    $$N = (n+2)^2 - 100$$
    To solve for N, the easiest approach is to solve for n first. Since we have two expressions for N, then
    $$(n+2)^2 - 100 = n^2 + 92$$
    $$n^2 + 4n + 4 - 100 = n^2 + 92$$
    $$4n - 96 = 92$$
    $$n = 47$$
    and thus
    $$N = (47)^2 + 92 = 2209 + 92 = 2301$$
    Therefore, Quentin has 2301 tiles.

    (b) Let B be the total number of blocks in Rufus' pile.
    When Quentin tries to make a cube with 8 blocks along an edge, he is 24 blocks short, and so he must have taken $8^3 - 24 = 488$ blocks.
    Let r be the edge length of the cube that Rufus makes with his portion of the blocks.
    Since the edge length of Rufus' original cube is r, he uses r^3 blocks.
    When Quentin and Rufus combine their blocks, they can make a cube of edge length $(r+2)$, and so have $(r+2)^3$ blocks in total.
    Thus,
    $$(r+2)^3 = r^3 + 488$$
    $$(r+2)^2(r+2) = r^3 + 488$$
    $$\left(r^2 + 4r + 4\right)(r+2) = r^3 + 488$$
    $$r^3 + 4r^2 + 4r + 2r^2 + 8r + 8 = r^3 + 488$$
    $$6r^2 + 12r - 480 = 0$$
    $$r^2 + 2r - 80 = 0$$
    $$(r+10)(r-8) = 0$$
    Since r is positive, then $r = 8$.
    Therefore, they have $(r+2)^3 = 10^3 = 1000$ blocks in total.

    Extension
    Let N be the total number of tiles that Quentin has, let x be the side length of the first square that Quentin tries to make, and let y be the side length of the second square that Quentin tries to make.
    As in (a), since Quentin has 92 extra tiles when he tries to make the first square and is 100 tiles short when he tries to make the second square, then

$$N = x^2 + 92$$

and

$$N = y^2 - 100$$

Again equating the values of N, we get

$$x^2 + 92 = y^2 - 106$$

$$y^2 - x^2 = 192$$

$$y^2 - 100 = x^2 + 92$$

$$y^2 - x^2 = 192$$

$$(y - x)(y + x) = 192$$

So we want to find the number of solutions to this equation where x and y are both integers with $y > x$. Thus, $y - x$ and $y + x$ are integers whose product is 192. Note also that $y + x$ is bigger than $y - x$. Now, $192 = 2(96) = 2(16)(6) = 2^6 3^1$.

The divisors of 192 are

1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 192

We should verify that each possibility gives an allowable value for N:

$y - x$	$y + x$	y	x	N
1	192			
2	96	49	47	2301
3	64			
4	48	26	22	584
6	32	19	13	269
8	24	16	8	164
12	16	14	2	104

(Notice that $y = \frac{1}{2}\left[(y - x) + (y + x)\right]$ so these equations are not too hard to solve!)
In the cases where $y - x$ is 1 and 3, y is not an integer, which is not allowed.
Therefore, there are 5 possible values for the number of tiles that Quentin has.

Comment
It is also possible to set up the equations in the following fashion

$$n - 92 = x^2$$

$$n + 100 = (x + p)^2$$

Subtracting the first equation from the second, we have

$$192 = (x + p)^2 - x^2 = 2xp + p^2 = p(2x + p)$$

The analysis continues in the same fashion as the first solution in that we must write $192 = 3 \cdot 2^6$ and consider divisors p and $2x + p$, where p is the smallest of the two divisors.

2. (a) *Solution 1*

First we ask the question: When does a player have a winning move? Since to win a player must remove the last coin, then a player has a winning move when he or she is choosing from a position with coins in only one pile (and a second empty pile).

So for Yolanda to win, she wants to ensure that she will always be passed an empty pile and a non-empty pile at some point. How can she force Xavier to pass her an empty pile? Xavier can only be forced to empty a pile if he receives two piles both of which have 1 coin (otherwise, he could reduce, but not empty, one of the piles).
So if Yolanda chooses from piles with 1 and 3 coins or 1 and 2 coins, then she can pass back piles with 1 and 1 coins, and be sure to win.

Thus, Xavier does not want to initially remove 2 coins from one pile, otherwise Yolanda can follow her strategy above. Also, Xavier does not want to remove 3 coins from one pile, or Yolanda can immediately win by removing the other 3 coins.

So Xavier should start by removing 1 coin, and passing piles with 2 and 3 coins to Yolanda. She does not want to pass an empty pile or a pile with 1 coin in it to Xavier (or he can use *her* strategy from above), so she removes 1 coin from the larger pile, and passes back 2 and 2 coins. Xavier is then forced to empty one pile, or reduce one pile to 1 coin, and so Yolanda can then guarantee that she wins.

Therefore, in all cases, Yolanda can guarantee that she wins.

Solution 2
Yolanda will always win the game if she can guarantee that at some point when it is her turn to choose that she is selecting coins from just one pile. If she is selecting coins from just one pile, she will win the game by removing all of the coins from that pile.

She can guarantee that this will happen by duplicating Xavier's move only in the other pile. Thus, if Xavier takes 1, 2 or 3 coins, then Yolanda will take the same number of coins from the other pile. This strategy, on Yolanda's part, will mean that Xavier will always empty one pile first, and thus guarantee that Yolanda will win.

Solution 3
If Yolanda can ensure that she passes two equal piles to Xavier, then Xavier can never win, because he can never empty the last pile (he'll always have two non-empty piles).
So if Xavier reduces to 2 and 3 coins, Yolanda passes back 2 and 2 coins.
If Xavier reduces to 1 and 3 coins, Yolanda passes back 1 and 1 coins.
If Xavier reduces to 0 and 3 coins, Yolanda can immediately win by removing the last 3 coins.
From 1 and 1 coins, Xavier must reduce to 1 and 0 coins, and so Yolanda wins.
From 2 and 2 coins, Xavier must reduce to 1 and 2 coins (allowing Yolanda to pass back 1 and 1 coins) or to 0 and 2 coins, allowing Yolanda to win immediately.
Therefore, Yolanda can always win by following an "equalizing" strategy.

(b) In part (a), we saw that Yolanda always won the game if she could guarantee that Xavier was choosing when there were two piles with an equal number of coins in each pile.

Starting with piles of 1, 2 and 3 coins, Yolanda can always win, because she can always after her first turn give two equal piles (and an empty third pile) back to Xavier. We see this by examining the possibilities:

Xavier's first move	Yolanda's first move
0,2,3	0,2,2
1,1,3	1,1,0
1,0,3	1,0,1
1,2,2	0,2,2
1,2,1	1,0,1
1,2,0	1,1,0

In any of these cases, Yolanda can be sure to win by following her "equalizing" strategy from part (a).
So Yolanda's strategy is to create two equal piles (and a third empty pile) after her first turn, and so force Xavier to lose, using her strategy from (a).

Extension

In part (b), we saw that if Xavier chooses first from three piles with 1, 2 and 3 coins, then Yolanda can always win.
In part (a), we saw that if Xavier chooses first from two piles with an equal number of coins, then Yolanda can again always win.
So on his first move, Xavier does not want to create two equal piles (eg. 2,4,4 or 2,2,5 etc.), otherwise Yolanda would remove the third unequal pile and Xavier would then be choosing on his second turn from two equal piles.
Similarly, Xavier does not want to create a situation where Yolanda can reduce immediately to 1,2,3, otherwise Yolanda will win by following the strategy from (b).

So we consider the possible first moves for Xavier:

Xavier's first move	Yolanda's first move	Winner
2,4,4	0,4,4	Yolanda
2,4,3	2,1,3	Yolanda
2,4,2	2,0,2	Yolanda
2,4,1	2,3,1	Yolanda
2,4,0	2,2,0	Yolanda
2,3,5	2,3,1	Yolanda
2,2,5	2,2,0	Yolanda
2,1,5	2,1,3	Yolanda
2,0,5	2,0,2	Yolanda
1,4,5	??	??
0,4,5	0,4,4	Yolanda

So if Xavier makes any move other than to 1,4,5, Yolanda will win by following the correct strategy.

What if Xavier moves to 1,4,5? There are then 10 possible moves for Yolanda. As above, if Yolanda makes her first move to 0,4,5 or 1,1,5 or 1,0,5 or 1,4,4 or 1,4,1 or 1,4,0, then Xavier can reduce to two equal piles. If Yolanda reduces to 1,3,5 or 1,2,5 or 1,4,3 or 1,4,2, then Xavier can reduce to some ordering of 1,2,3, and so Xavier can win. Therefore, Xavier can win by reducing first to 1,4,5, and then to either two equal piles or some ordering of 1,2,3, and then following Yolanda's strategy from (a) or (b).

3. We slice the two solids with a vertical plane through the vertex of the cone and highest point of the sphere. The cross-sections we get then are an isosceles triangle with a height of 10 cm and a base of 10 cm, and a circle of diameter 10 cm.

We want to find the height, h, that produces equal cross-sectional areas. Since the formula for the area of a circle is πr^2 and we want to find h, we need to find a relationship between h and r for the cone and for the sphere.

Case 1 – Cone
From V, the vertex of the cone, we draw in the principal axis of the cone as shown. It is clear by symmetry that triangle VPQ is similar to triangle VRS.
Thus,

$$\frac{10}{5} = \frac{10-h}{r}$$
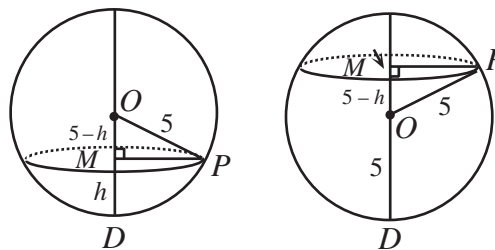$$2r = 10 - h$$
$$r = \tfrac{1}{2}(10-h)$$
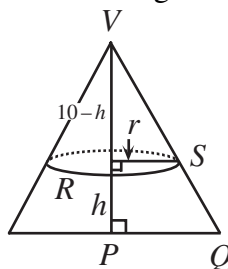
Case 2 – Sphere
From O, the centre of the sphere, we draw a line through the centre of the cross-sectional circle. This line is perpendicular to the circle. We have $OP = 5$ (the radius of the circle), $DO = 5, DM = h$, and $MP = r$.
We need an expression for r – the answer depends on whether h is bigger or smaller than 5.

If h is bigger than or equal to 5, then $OM = h - 5$. If h is smaller than 5, then $OM = 5 - h$.

Thus, by Pythagoras,
$$r = \sqrt{OP^2 - OM^2} = \sqrt{5^2 - (5-h)^2} = \sqrt{25 - \left(h^2 - 10h + 25\right)} = \sqrt{10h - h^2}$$

(Notice that even if $OM = h - 5$, then we get the same answer, since $(h-5)^2 = (5-h)^2$.)

So, we compare the areas of the cross-sectional circles to solve for h:

$$\pi\left[\tfrac{1}{2}(10-h)\right]^2 = \pi\left[\sqrt{10h-h^2}\right]^2$$

$$(10-h)^2 = 4\left(10h-h^2\right)$$

$$100-20h+h^2 = 40h-4h^2$$

$$5h^2-60h+100 = 0$$

$$h^2-12h+20 = 0$$

$$(h-10)(h-2) = 0$$

and therefore, $h=10$ or $h=2$.

(Note that $h=10$ gives a horizontal plane that just passes through the vertex of the cone and is tangent to the top of the sphere.)

Therefore, the height is 10 cm or 2 cm.

Extension

To avoid fractions, we start by letting $d=2R$, so the cone has radius R and height R.
As above, we slice the two solids with a vertical plane.

In the cone, at a height of h, we have $\dfrac{R-h}{R} = \dfrac{r}{R}$ or $r=R-h$ as in the previous set-up.
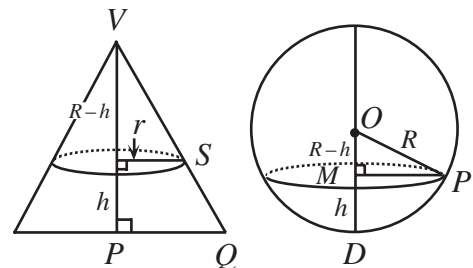
In the sphere, at a height of h we have

$$r = \sqrt{OP^2-OM^2} = \sqrt{R^2-(R-h)^2} = \sqrt{R^2-\left(h^2-2Rh+R^2\right)} = \sqrt{2hR-h^2}$$

Note that $h \le R$ since the cone has height R.
Therefore, the sum of the areas of the two circular cross-sections is

$$\pi\left(\sqrt{2hR-h^2}\right)^2 + \pi(R-h)^2$$

$$= \pi\left(2hR-h^2\right) + \pi\left(R^2-2hR+h^2\right)$$

$$= \pi R^2$$

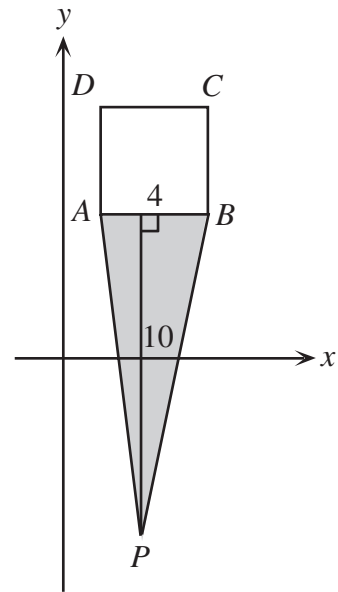which is a constant, since R is a constant.

4. (a) *Solution*
   The visible area of $P(2,-6)$ is the area of $\triangle ABP$.
   $\triangle ABP$ has base AB of length 4, and its height is the distance from AB to the point P, which is 10, since AB is parallel to the x-axis.
   Thus, the area of $\triangle ABP$ is $\frac{1}{2}bh = \frac{1}{2}(4)(10) = 20$ square units, ie. the visible area of P is 20 square units.
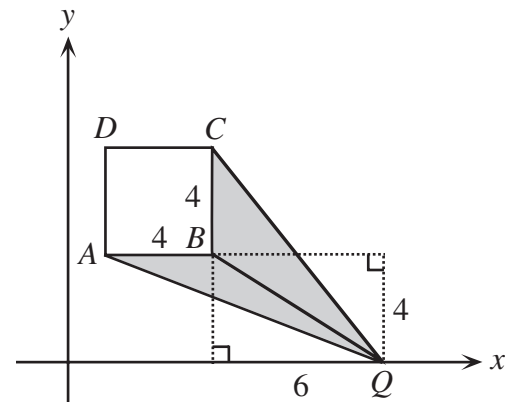
   (b) *Solution*
   The visible area of $Q(11,0)$ is the sum of the areas of $\triangle QBC$ and $\triangle QBA$.
   $\triangle QBC$ has base BC of length 4, and its height is the distance from Q to the line through B and C, which is 6.
   Thus, the area of $\triangle QBC$ is $\frac{1}{2}bh = \frac{1}{2}(4)(6) = 12$ square units.
   $\triangle QBA$ has base BA of length 4, and its height is the distance from Q to the line through B and A, which is also 4. Thus, the area of $\triangle QBA$ is $\frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$ square units.
   So the visible area of Q is the sum of these two areas, or 20 square units.

(c) **Solution**
From any point *P*, there are either 2 or 3 visible
vertices of the square.
We first consider those points *P* for which
there are 2 visible vertices.
Geometrically, these will be points which lie
"directly opposite" an edge of the square.
That is, they will be point *P* which
i)   lie below the square, with *x*-coordinate
     between 1 and 5, inclusive
ii) lie above the square, with *x*-coordinate
     between 1 and 5, inclusive
iii) lie to the left of the square, with *y*-
     coordinate between 4 and 8, inclusive
iv) lie to the right of the square, with *y*-
     coordinate between 4 and 8, inclusive
In each of these four cases, the visible area will
be a single triangle whose base is a side of the
square (that is, of length 4). For the visible
area of one of these points *P* to be 20, the
height of this triangle must be 10.

Thus, for those points in case (i), all points *P*
which lie 10 units below the square, and have
x-coordinates between 1 and 5 will be in the
20/20 set. In other words, the points *P* lying
on the line segment joining $(1,-6)$ to $(5,-6)$
will lie in the 20/20 set. (Notice that this
includes the point $(2,-6)$ from (a).) These are
all of the points in this region that lie in the
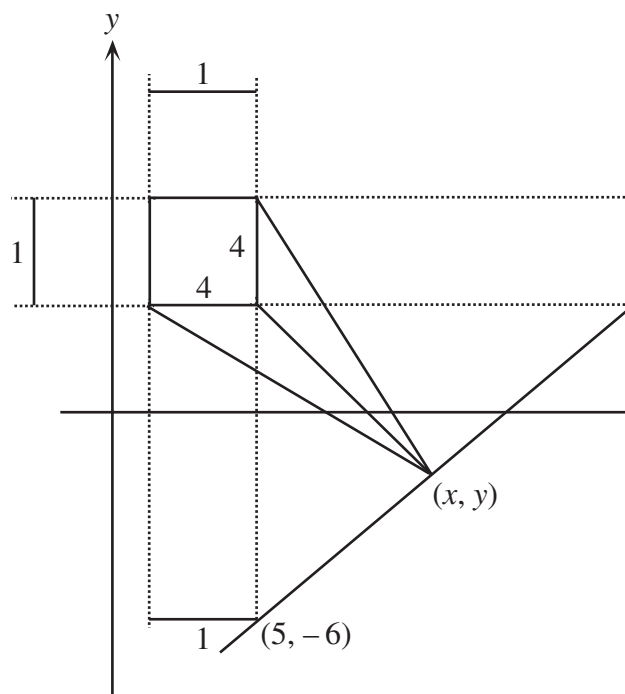20/20 set. So in this region, the 20/20 set has
length 4.
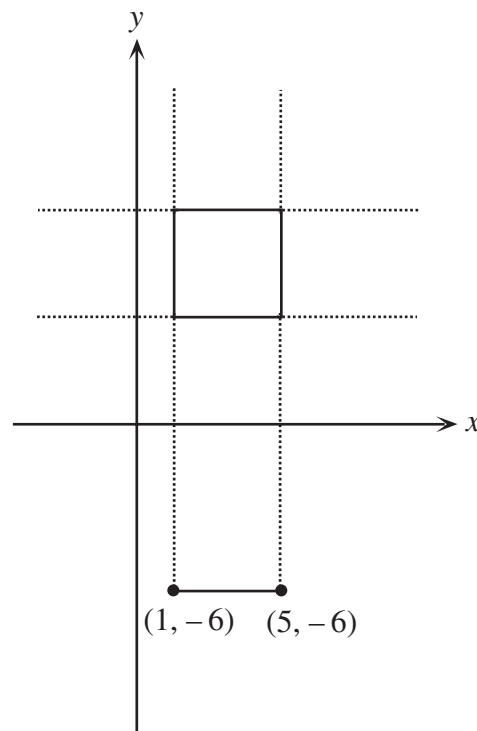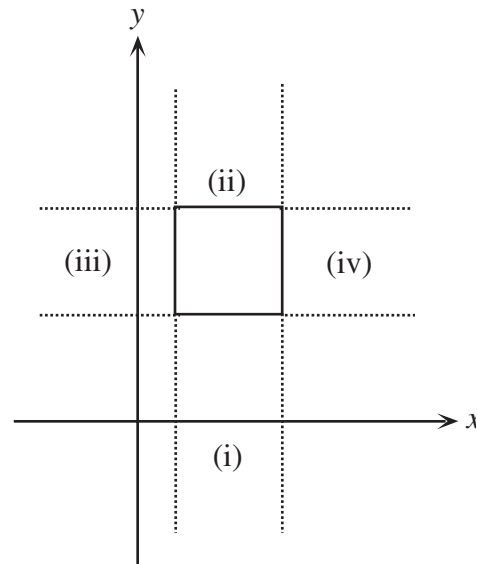
The other three regions will give the same result, by symmetry, so we have four line segments, each of length 4, in the 20/20 set thus far.

There are four more regions to consider – that is, the regions that do not lie directly opposite an edge of the square (for example, the region of points $P(x,y)$ where $x \geq 5$ and $y \leq 4$). By symmetry each of these regions should give us the same result.

So suppose that $P(x,y)$ in the region $x \geq 5$ and $y \leq 4$ has a visible area of 20. Then the sum of the areas of $\triangle PBC$ and $\triangle PBA$ is 20.
$\triangle PBC$ has a base of length 4 and a height of $x-5$.
$\triangle PBA$ has a base of length 4 and a height of $4-y$.

Therefore, for P to be in the 20/20 set, we have

$$\tfrac{1}{2}(4)(x-5) + \tfrac{1}{2}(4)(4-y) = 20$$
$$x - 5 + 4 - y = 10$$
$$y = x - 11$$

ie. P lies on the straight line of slope 1, $y = x - 11$. (Notice that this includes the point $(11,0)$ from (b).) So in this region, the points in the 20/20 set are the points on the line segment joining $(5,-6)$ to $(15,4)$, which has length $10\sqrt{2}$. (It is worth noting that the endpoints of this line segment are the endpoints of the line segments from regions (i) and (iv) above.)

We can argue by symmetry that the other sides in the required octagon will be as shown in the diagram.

So the 20/20 is a polygon (in fact, an octagon) with four sides of length 4 and four sides of length $10\sqrt{2}$. Therefore, the perimeter of the 20/20 set is $16 + 40\sqrt{2}$.
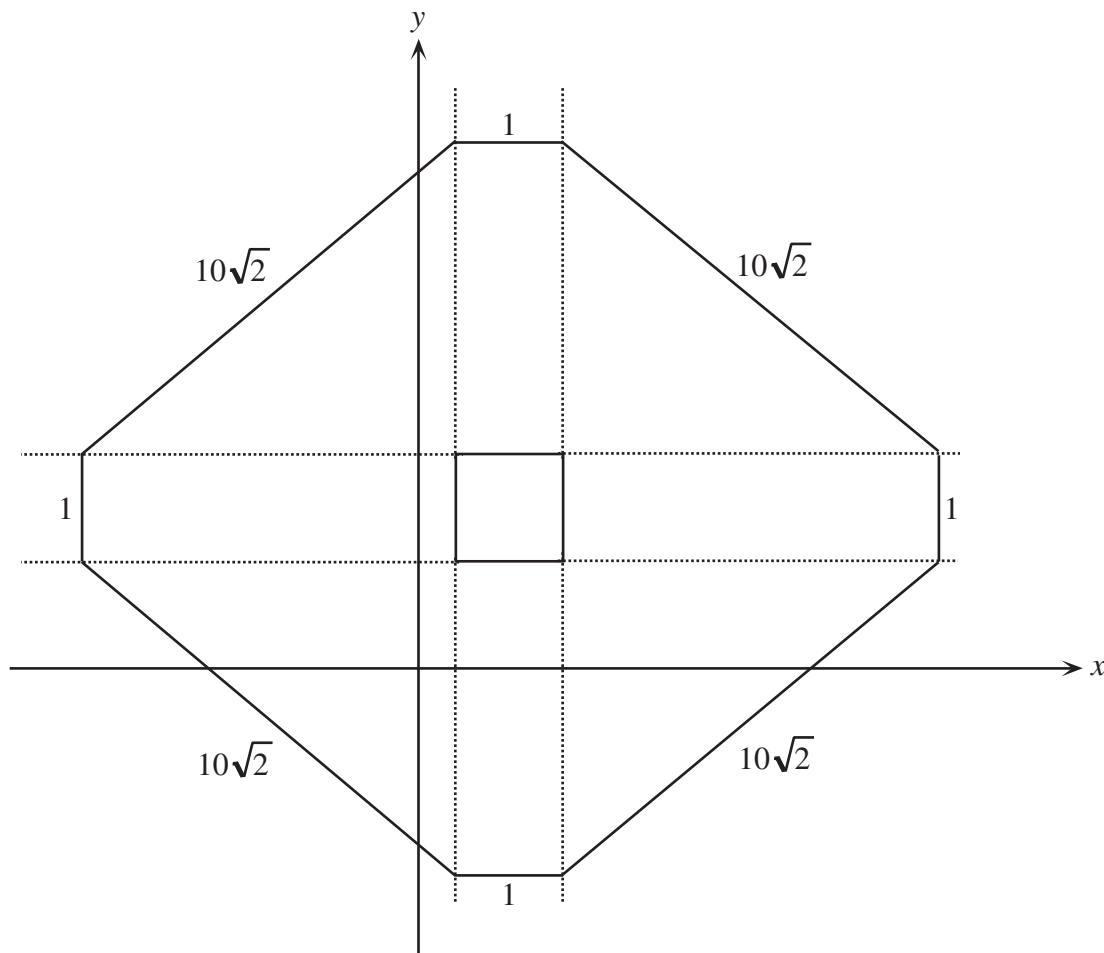
Extension

We approach this in an analogous manner to the 2-dimensional case.

Case 1: 4 visible vertices

Where will a point P lie so that it has exactly 4 visible vertices?

In this case, P will lie directly opposite one of the 6 faces of the cube.

The visible volume will be formed by one square-based pyramid.

The base of this pyramid will be a unit square, so since the volume of a pyramid is one-third times the area of the base times the height, then for a volume of 20, the height of the pyramid must be 60.

So the points P above one of the faces which have a visible volume of 20 are all those points 60 units above the faces. In other words, the points form a square (again, of side length 1) which is 60 units above the face. So the 20/20 set has six square faces each of area 1.

Case 2: 6 visible vertices

Where will a point P lie so that it has exactly 6 visible vertices?

In this case, the 6 visible vertices are the 6 vertices of two faces which share a common edge, and P lies in the region which is between the outer edges of these two faces, but not

directly above either face. We say that P lies above the edge. Since the cube has 12 edges, there will be 12 such regions.

Consider points P in one of these regions. The visible volume here will be formed by two square-based pyramids, each of which has a base which is a unit square. So the visible volume will be one-third times the area of the base (which is 1) times the sum of the two heights. Since the visible volume should be 20, the sum of the two heights will be 60.

The set of points above the edge which give a combined height of 60 for the two pyramids will form a rectangle which joins the edges of the two squares over the two adjacent faces. This rectangle will thus have one edge of length 1 and the other edge of length $60\sqrt{2}$ (since the long edge will form the hypotenuse of a right-angled triangle with two legs each of length 60).

(Why does this give us a portion of a plane? If we consider points P which lie in a cross-section which is perpendicular to the two adjacent faces, then we are asking for P to have sum of the heights to two perpendicular line segments to be 60, which gives a straight line as in (b). Sliding this cross-section across the faces gives us a segment of a plane.)

Thus, we have twelve rectangular faces, each of area $60\sqrt{2}$.

Case 3: 7 visible vertices

Where will a point P have 7 visible vertices?

P will have 7 visible vertices if it lies in a position that has not yet been considered. That is, P will in essence be above one of the 8 vertices of the cube. (This seems sensible since the 20/20 set currently has 8 holes in it!)

This time, the visible volume is made up of three square-based pyramids, so we want points P so the sum of the heights of these three pyramids is 60, as before.

This region will again form part of a plane, and it should connect to the three rectangles already formed over the three edges that meet at the vertex in question (since at the edge of each of these rectangles, we can consider having a third pyramid of height 0).

So this segment of the 20/20 set is an equilateral triangle of side length $60\sqrt{2}$.

To find the area of this triangle, we join the top vertex to the midpoint of the base, so each half of the triangle is a 30-60-90 right-angled with a leg of length $30\sqrt{2}$ opposite the $30°$ angle. Thus, the height of the triangle is $\sqrt{3}\left(30\sqrt{2}\right) = 30\sqrt{6}$, and so the area of the whole equilateral triangle is $\frac{1}{2}\left(60\sqrt{2}\right)\left(30\sqrt{6}\right) = 900\sqrt{12} = 1800\sqrt{3}$.
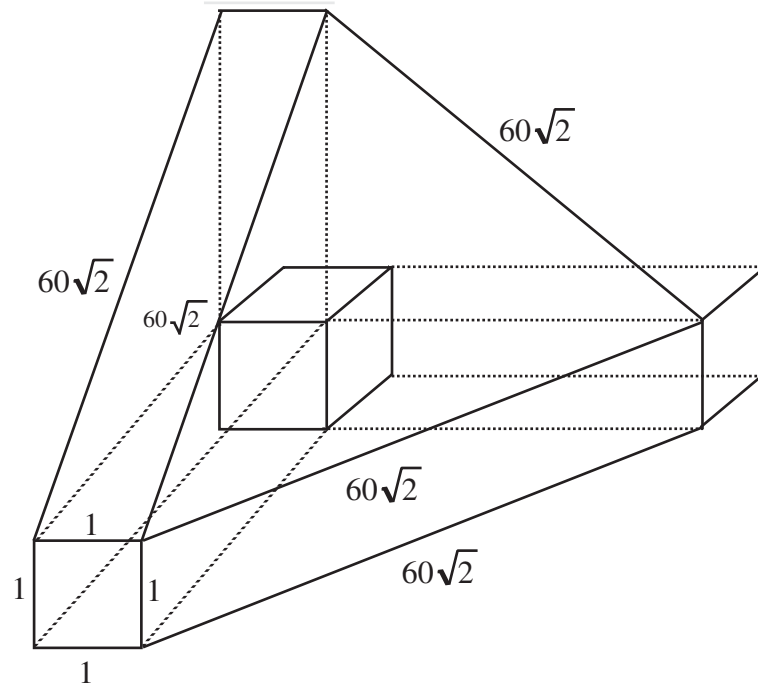
A partial sketch of the desired region.
Therefore, the total surface area of the 20/20 set is

$$6(1) + 12(60\sqrt{2}) + 8(1800\sqrt{3}) = 6 + 720\sqrt{2} + 14400\sqrt{3}$$