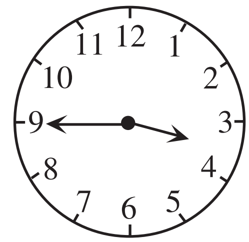
2003 Galois Contest *(Grade 10)*

Wednesday, April 16, 2003

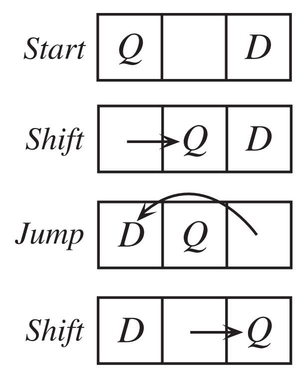
1. (a) The *sum of the squares* of 5 consecutive positive integers is 1815. What is the largest of these integers?
   (b) Show that the sum of the squares of any 5 consecutive integers is divisible by 5.

2. Professor Cuckoo mistakenly thinks that the angle between the minute hand and the hour hand of a clock at 3:45 is 180°.
   (a) Through how many degrees does the hour hand pass as the time changes from 3:00 p.m. to 3:45 p.m.?
   (b) Show that the Professor is wrong by determining the exact angle between the hands of a clock at 3:45.
   (c) At what time between 3:00 and 4:00 will the angle between the hands be 180°?

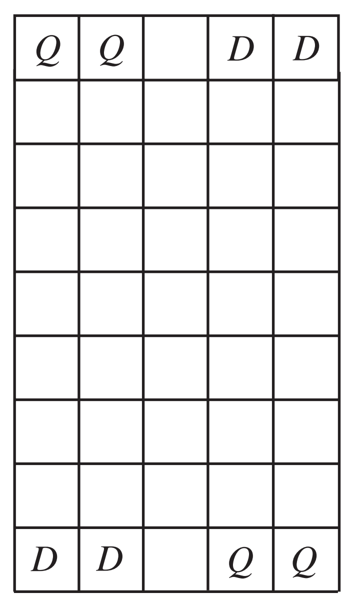


3. In the game "Switch", the goal is to make the dimes (D) and quarters (Q) switch spots. The starting position of the game with 1 quarter and 1 dime is shown below. Allowable moves are:
   (i) If there is a vacant spot beside a coin then you may *shift* to that space.
   (ii) You may *jump* a quarter with a dime or a dime with a quarter if the space on the other side is free.

   The game shown in the diagram takes three moves.



   (a) Complete the diagram to demonstrate how the game of "Switch" that starts with 2 quarters and 2 dimes can be played in 8 moves.



   (b) By considering the number of required *shifts* and *jumps*, explain why the game with 3 quarters and 3 dimes cannot be played in fewer than 15 moves.

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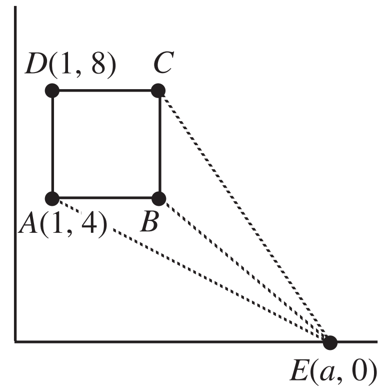
4. In the diagram, $ABCD$ is a square and the coordinates of A and D are as shown.

   (a) The point $E(a, 0)$ is on the x-axis so that the triangles CBE and ABE lie entirely outside the square $ABCD$. For what value of a is the sum of the areas of triangles CBE and ABE equal to the area of square $ABCD$?

   (b) The point F is on the line passing through the points $M(6, -1)$ and $N(12, 2)$ so that the triangles CBF and ABF lie entirely outside the square $ABCD$. Determine the coordinates of the point F if the sum of the areas of triangle CBF and ABF equals the area of square $ABCD$.



Extensions (Attempt these only when you have completed as much as possible of the four main problems.)

Extension to Problem 1:
The number 1815 is also the sum of 5 consecutive positive integers. Find the next number larger than 1815 which is the sum of 5 consecutive integers and also the sum of the squares of 5 consecutive integers.

Extension to Problem 2:
The assumption might be made that there are 24 times during any 12 hour period when the angle between the hour hand and the minute hand is 90°. This is not the case. Determine the actual number of times that the angle between the hour and minute hands is 90°.

Extension to Problem 3:
Explain why the game with n quarters and n dimes cannot be played in fewer than $n(n+2)$ moves.

Extension to Problem 4:
Find the set of all points $P(x, y)$ which satisfy the conditions that the triangles CBP and ABP lie entirely outside the square $ABCD$ and the sum of the areas of triangles CBP and ABP equals the area of square $ABCD$.