Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2003 Solutions
Fryer Contest (Grade 9)

1. (a) *Solution 1*

The average (mean) is equal to the sum of all of the marks, divided by the total number of marks.

Since we know already that there are 32 students (we can check this by looking at the graph), then the average is

$$\frac{1(10)+2(30)+2(40)+1(50)+4(60)+6(70)+9(80)+4(90)+3(100)}{32}=\frac{2240}{32}=70$$

Therefore, the average mark was 70.

Solution 2

The average (mean) is equal to the sum of all of the marks, divided by the total number of marks.

Using the bar graph, we can list out all of the marks:

10, 30, 30, 40, 40, 50, 60, 60, 60, 60, 70, 70, 70, 70, 70, 70, 80, 80, 80, 80, 80, 80, 80, 80, 80, 90, 90, 90, 90, 100, 100, 100.

Adding these up using a calculator and dividing by 32, we find that the average mark is 70.

(b) After his first 6 tests, since Paul's average is 86, then has gotten a total of $6(86)=516$ marks.

After getting 100 on his seventh test, Paul has gotten a total of $516+100=616$ marks, so his new average is $\frac{616}{7}=88$.

(c) *Solution 1*

If Mary gets 100, her average becomes 90.

If Mary gets 70, her average becomes 87.

So a difference of 30 marks on the test gives a difference of 3 marks in the average.

Since her average is her total number of marks divided by her total number of tests, and a difference of 30 in the total number of marks makes a difference of 3 in her average, then she will have written $\frac{30}{3}=10$ tests.

Solution 2

Suppose that after the next test, Mary has written n tests.

If her average after getting 100 on the next test is 90, then Mary has earned $90n$ marks in total after the first n tests, and so $90n-100$ before she writes the nth test.

If her average after getting 70 on the next test is 87, then Mary has gotten $87n$ marks in total after the nth test, and so she will have earned $87n-70$ marks before the next test.

Therefore, since the number of marks before her next test is the same in either case,

$$
\begin{aligned}
87n - 70 &= 90n - 100 \\
30 &= 3n \\
n &= 10
\end{aligned}
$$

So Mary will have written 10 tests.

Extension

We start by using the given information to try to figure out some more things about the marks of the 32 students.

Since the median mark is the "middle mark" in a list of marks which is increasing, then there at least 16 students who have marks that are at least 80.

Since the difference between the highest and lowest marks is 40, and there are students who got at least 80, then the lowest mark in the class cannot be lower than 40.

Since the average mark in the class is 58, then the total number of marks is $32(58) = 1856$.

So what does this tell us?

Since at least 16 students got at least 80, then this accounts for at least 1280 marks, leaving $1856 - 1280 = 576$ marks for the remaining 16 students.

But the lowest possible mark in the class was 40, so these remaining 16 students got at least 40 each, and so got at least $16(40) = 640$ marks in total!

So we have an inconsistency in the data.

Thus, the teacher made a calculation error.

(There is a variety of different ways of reaching this same conclusion. As before, we know that 16 students will have a mark of at least 80, which accounts for 1280 marks. By the same reasoning, the other 16 students would account for the other 576 marks. The average for these 16 students is thus about 34, which implies that at least one of these lower students must have a mark of 34 or lower. This now contradicts the statement that the range is 40 since $80 - 34 > 40$.)

2. (a) *Solution 1*

If Xavier goes first and calls 4, then on her turn Yolanda can call any number from 5 to 14, since her number has to be from 1 to 10 greater than Xavier's.

But if Yolanda calls a number from 5 to 14, then Xavier can call 15 on his next turn, since 15 is from 1 to 10 bigger than any of the possible numbers that Yolanda can call.

So Xavier can call 15 on his second turn no matter what Yolanda calls, and is thus always guaranteed to win.

Solution 2

If Xavier goes first and calls 4, then Yolanda will call a number of the form $4+n$ where n is a whole number between 1 and 10.

On his second turn, Xavier can call 15 (and thus win) if the difference between 15 and $4+n$ is between 1 and 10. But $15-(4+n)=11-n$ and since n is between 1 and 10, then $11-n$ is also between 1 and 10, so Xavier can call 15.

Therefore, Xavier's winning strategy is to call 15 on his second turn.

(b) In (a), we saw that if Xavier calls 4, then he can guarantee that he can call 15.
Using the same argument, shifting all of the numbers up, to guarantee that he can call 50, he should call 39 on his previous turn.
(In this case, Yolanda can call any whole number from 40 to 49, and in any of these cases Xavier can call 50, since 50 is no more than 10 greater than any of these numbers.)
In a similar way, to guarantee that he can call 39, he should call 28 on his previous turn, which he can do for the same reasons as above.
To guarantee that he can call 28, he should call 17 on his previous turn.
To guarantee that he can call 17, he should call 6 on his previous turn, which could be his first turn.
Therefore, Xavier's winning strategy is to call 6 on his first turn, 17 on his second turn, 28 on his third turn, 39 on his fourth turn, and 50 on his last turn.
At each step, we are using the fact that Xavier can guarantee that his number on one turn is 11 greater than his number on his previous turn. This is because Yolanda adds 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 to his previous number, and he can then correspondingly add 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1 to her number, for a total of 11 in each case.

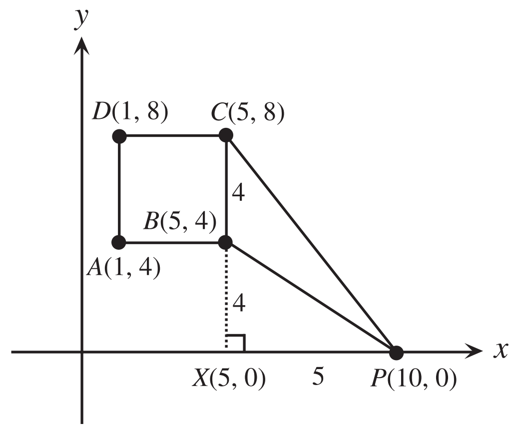
Extension

In (b), we discovered that Xavier can always guarantee that the difference between his numbers on two successive turns is 11.
In fact, Yolanda can do the same thing, using exactly the same strategy as Xavier did.
If the target number is between 1 and 9, then Xavier will win on his very first turn by calling that number.
If the target number is then 11 greater than a number between 1 and 9, Xavier will win as in (b). Thus, Xavier wins for 12 through 20.
What about 10 and 11? In each of these cases, Yolanda can win by choosing 10 or 11 on her first turn, which she can do for any initial choice of Xavier's, since he chooses a number between 1 and 9.
Therefore, Yolanda will also win for 21 and 22, and so also for 32 and 33, and so on.
Since either Yolanda or Xavier can repeat their strategy as many times as they want, then Xavier can ensure that he wins if the target number is a multiple of 11 more than one of 1 through 9.

Similarly, Yolanda can ensure that she wins if the target number is a multiple of 11 more than 10 or 11, ie. if the target number is a multiple of 11, or 1 less than a multiple of 11.

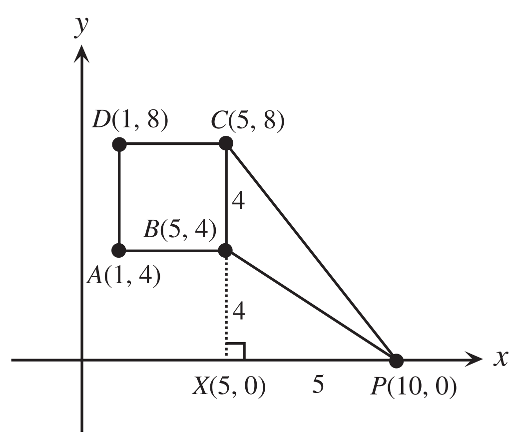
3. (a) *Solution 1*
Since $ABCD$ is a square and AD has side length 4, then each of the sides of $ABCD$ has length 4. We can also conclude that B has coordinates $(5,4)$ and C has coordinates $(5,8)$. (Since AD is parallel to the y-axis, then AB is parallel to the x-axis.)

If we turn ΔPBC on its side, then its we see that its base is BC which has length 4. Also the height of the triangle is the vertical distance from the line BC to P, which is 5. (We can see this by extending the line CB to a point X on the x-axis. Then X has coordinates $(5,0)$ and PX, which has length 5, is perpendicular to CB.)
Therefore, the area of ΔPBC is $\frac{1}{2}bh = \frac{1}{2}(4)(5) = 10$.



Solution 2
Since $ABCD$ is a square and AD has side length 4, then all of the sides of $ABCD$ have length 4. We can also conclude that B has coordinates $(5,4)$ and C has coordinates $(5,8)$. (Since AD is parallel to the y-axis, then AB is parallel to the x-axis.)

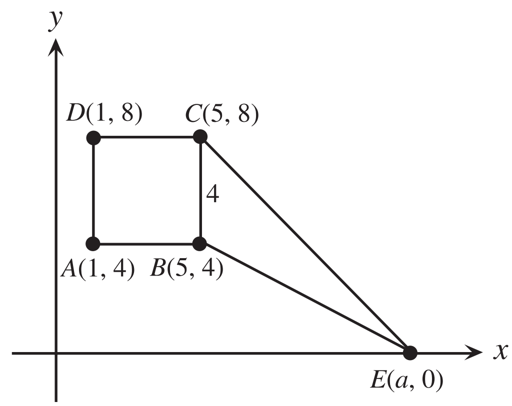
Extend CB down to a point X on the x-axis.
Point X has coordinates $(5,0)$.
Then the area of ΔPBC is the difference between the areas of ΔPCX and ΔPBX.
ΔPCX has base PX of length 5 and height CX of length 8.
ΔPBX has base PX of length 5 and height BX of length 4.
Therefore, the area of ΔPBC is
$$\tfrac{1}{2}(4)(10) - \tfrac{1}{2}(4)(5) = 10$$
as required.



(b) *Solution 1*
Since triangle CBE lies entirely outside square $ABCD$, then the point E must be "to the right" of the square, ie. a must be at least 5.

Also, we need to know the area of the square. Since the side length of the square is 4, its area is 16.

Thus if we turn ΔCBE on its side, then its we see that its base is BC which has length 4. Also the height of the triangle is the vertical distance from the line BC to E, which is $a-5$ since E has coordinates $(a,0)$ and BC is part of the line $x=5$.
Therefore, since the area of ΔCBE is equal to the area of the square,

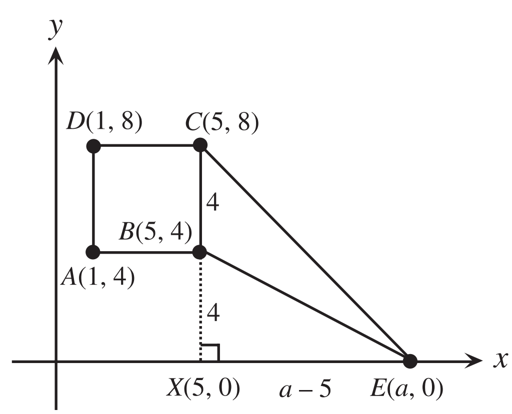


$$\begin{aligned}\tfrac{1}{2}(4)(a-5)&=16\\ 2a-10&=16\\ 2a&=26\\ a&=13\end{aligned}$$

Thus, $a=13$.
(It is easy to verify that if $a=13$, then the height of the triangle is 8 and its base is 4, giving an area of 16.)

Solution 2
Since triangle CBE lies entirely outside square $ABCD$, then the point E must be "to the right" of the square, ie. a must be at least 5.
Also, we need to know the area of the square. Since the side length of the square is 4, its area is 16.

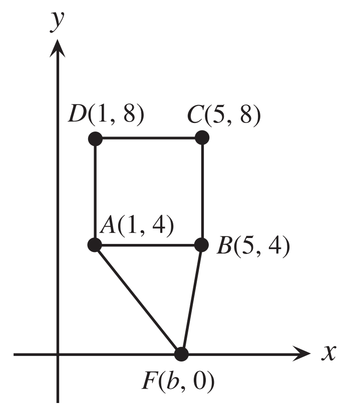
Extend CB down to a point X on the x-axis. Point X has coordinates $(5,0)$.
Then the area of ΔCBE is the difference between the areas of ΔCXE and ΔBXE.
ΔCXE has base EX of length $a-5$ and height CX of length 8.
ΔBXE has base EX of length $a-5$ and height BX of length 4.
Therefore, since the area of ΔCBE is equal to the area of the square,



$$\begin{aligned}\tfrac{1}{2}(a-5)(8)-\tfrac{1}{2}(a-5)(4)&=16\\ 4(a-5)-2(a-5)&=16\\ a-5&=8\\ a&=13\end{aligned}$$

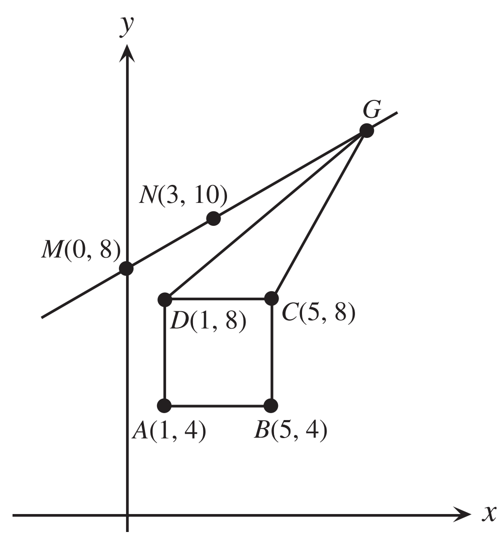
Thus, $a = 13$, as required.

(c) Suppose that F has coordinates $(b,0)$.
Then triangle ABF has base AB of length 4.
The height of triangle ABF is the vertical distance from F to the line AB, which is always 4, no matter where F is.
Thus, the area of triangle ABF is always $\frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$, which is not equal to the area of the square.



Extension

Since triangle DCG lies entirely outside the square, then G is "above" the line through D and C, ie. the y-coordinate of G is at least 8.
Since the area of triangle DCG is equal to the area of the square, then the area of triangle DCG is 16.
Now triangle DCB has base DC, which has length 4, so $\frac{1}{2}bh = \frac{1}{2}(4)h = 16$ or $h = 8$.
Since the height of triangle DCG is 8, then G has y-coordinate 16, since both D and C have y-coordinate 8.
So we must find the point on the line through $M(0,8)$ and $N(3,10)$ which has y-coordinate 16.
To get from M to N, we go 3 to the right and up 2.
To get from N to G, we go up 6, so we must go 9 to the right.
Therefore, G has coordinates $G(12,16)$.



4. (a) The best approach here is to list the numbers directly. The possible totals are, from smallest to largest:

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

There 15 possible totals, and their sum (that is, the power-sum) is 8888.

(b) *Solution 1*

First, we consider the numbers that are sums of 1 or more of the numbers from $\{1,10,100,1000\}$. In (a), we saw that the sum of these numbers is 8888.

What happens when we consider numbers that are sums of 1 or more of the numbers from $\{1{,}10{,}100{,}1000{,}10\,000\}$? When we do this, we obtain all 15 of the numbers from the previous paragraph, we obtain the 15 numbers obtained by adding 10 000 to all of the numbers from the previous paragraph, and also the number 10 000. (Either our sum does not include 10 000 as a term, or it does; if it doesn't, it must be one of the numbers from (a); if it does, it could be 10 000 on its own, or it could be 10 000 plus one of the numbers from (a).)
Therefore, we have $15+15+1=31$ numbers in total, whose sum is

$$8888+\left[8888+15(10\,000)\right]+10\,000=2(8888)+160\,000=2(88\,888)=177\,776$$

What happens when we consider the numbers that are sums of 1 or more of the numbers from $\{1{,}10{,}100{,}1000{,}10\,000{,}100\,000\}$? When we do this, we obtain all 31 of the numbers from (a), we obtain the 31 numbers obtained by adding 100 000 to all of the numbers from (a), and also the number 100 000.
Therefore, we have $31+31+1=63$ numbers in total, whose sum is

$$\begin{aligned}177\,776+\left[177\,776+31(100\,000)\right]+100\,000&=2(177\,776)+3\,200\,000\\&=3\,555\,552\end{aligned}$$

What happens when we add 1 000 000 to the set? We then obtain, as before, $63+63+1=127$ numbers in total, whose sum is

$$\begin{aligned}3\,555\,552+\left[3\,555\,552+63(1\,000\,000)\right]+1\,000\,000&=2(3\,555\,552)+64\,000\,000\\&=71\,111\,104\end{aligned}$$

Therefore, the power sum is 71 111 104.

Solution 2
There are seven numbers in the given set. When we are considering sums of one or more numbers from the set, each of the seven numbers in the set is either part of the sum, or not part of the sum. So there are two choices ("in" or "out") for each of the 7 elements.

So we can proceed by first choosing the elements we want to add up, and then adding them up. Since for each of the two possibilities for the "1" (ie. chosen or not chosen), there are two possibilities for the "10", and there are two possibilities for the "100", and so on. In total, there will be $2\times2\times2\times2\times2\times2\times2=2^7=128$ ways of choosing elements. Notice that this includes the possibility of choosing no elements at all (since we could choose not to select each of the seven elements).

So there are 128 possible sums (including the sum which doesn't add up any numbers at all!).

In how many of these sums is the “1” chosen? If the “1” is chosen, then there are still 2 possibilities for each of the remaining six elements (either chosen or not chosen), so there are $2^6 = 64$ sums with the 1 included, so the 1 contributes 64 to the power-sum.
In how many of these sums is the “10” chosen? Using exactly the same reasoning, there are 64 sums which include the 10, so the 10 contributes 640 to the power-sum.

Extending this reasoning, each of the 7 elements will contribute to 64 of the sums. (Note that including the “empty” sum doesn’t affect the power-sum.)
Therefore, the power sum is

$$64(1+10+100+1000+10000+100000+1000000)$$
$$= 64(1111111)$$
$$= 71111104$$

Extension

We start looking at small numbers to see if we can see a pattern.
Using the numbers 1, 2, 3, we can form the sums

$$1, 2, 3, 1+3=4, 1+3=4, 2+3=5, 1+2+3=6$$

If we include the 6, we can obtain all of these sums, as well as 6 plus each of these sums. In other words, we obtain each of the numbers 1 through 12 as totals.
Then including the 12, we can obtain 13 through 24, so we now have each of 1 to 24 as totals.
Including the 24, we obtain all numbers up to 48.
Including the 48, we obtain all numbers up to 96.
Including the 96, we obtain all numbers up to 192.
Therefore, there are 192 different totals possible.
(We can check as well that the sum of the elements in the original set is 192.)