Canadian
Mathematics
Competition

An activity of The Centre for Education
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2003 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

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2003 Cayley Contest Solutions

1. Evaluating,
$$\frac{3-(-3)}{2-1}=\frac{6}{1}=6.$$
ANSWER: (D)

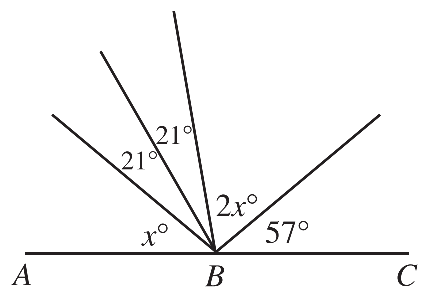
2. $17^2-15^2=289-225=64=8^2$. ANSWER: (A)

3. Since $42=6\times 7$, the only possibility for the correct answer is that 42 is divisible by 7. (We can check that each of the remaining possibilities is not true.)
ANSWER: (D)

4. Since 25% of the number is 5 times 5% of that number, then 25% of the number is $5(8)=40$.
ANSWER: (A)

5. We could use a calculator to determine the value of the expression and then round to the nearest integer. Alternatively, we can calculate the value of the expression by hand:
$$\tfrac{3}{2}\times\tfrac{4}{9}+\tfrac{7}{2}=\tfrac{12}{18}+\tfrac{7}{2}=\tfrac{12}{18}+\tfrac{63}{18}=\tfrac{75}{18}=\tfrac{25}{6}=4+\tfrac{1}{6}$$
Therefore, the closest integer is 4.
ANSWER: (B)

6. Since ABC is a straight line, the sum of the 5 angles is 180°, and so
$$\begin{aligned} x^\circ+21^\circ+21^\circ+2x^\circ+57^\circ &= 180^\circ \\ 3x+99 &= 180 \\ 3x &= 81 \\ x &= 27 \end{aligned}$$



ANSWER: (A)

7. *Solution 1*
In the top right quarter, we have no unknowns, so we can calculate the sum of the three numbers to be $13+17+45=75$. Therefore, the sum of the numbers (including the unknowns) in each of the other 3 quarters is 75 as well, and so
$$\begin{aligned} (z+28+8)+(x+19+50)+(y+3+63) &= 3(75) \\ x+y+z+36+69+66 &= 225 \\ x+y+z &= 54 \end{aligned}$$

Solution 2
In the top right quarter, we have no unknowns, so we can calculate the sum of the three numbers to be $13+17+45=75$.
In the top left quarter, we then have $z+28+8=75$ and so $z=39$.

Similarly, $x = 6$ and $y = 9$.
Therefore, $x + y + z = 6 + 9 + 39 = 54$. ANSWER: (C)

8. An equilateral triangle with a side length of 20 must have a perimeter of 60.
A square with a perimeter of 60 must have a side length of 15.
A square with a side length of 15 must have an area of 225.

ANSWER: (C)

9. *Solution 1*
Taking the reciprocal of both sides and then solving,

$$\begin{aligned} \frac{1}{x+\frac{1}{5}} &= \frac{5}{3} \\ x + \tfrac{1}{5} &= \tfrac{3}{5} \\ x &= \tfrac{2}{5} \end{aligned}$$

Solution 2
Cross-multiplying,

$$\begin{aligned} \frac{1}{x+\frac{1}{5}} &= \frac{5}{3} \\ 5\left(x + \tfrac{1}{5}\right) &= 3 \\ 5x + 1 &= 3 \\ x &= \tfrac{2}{5} \end{aligned}$$

ANSWER: (A)

10. *Solution 1*
When $\frac{5}{8}$ of the players are girls then $\frac{3}{8}$ of the players will be boys. Since the number of boys playing is 6 (and does not change), then after the additional girls join, there must be 16 players in total for $\frac{3}{8}$ of the players to be boys. Since there were 8 players initially, then 8 additional girls must have joined the game.

Solution 2
Let the number of additional girls be g.
Then

$$\begin{aligned} \frac{2+g}{8+g} &= \frac{5}{8} \\ 16 + 8g &= 40 + 5g \\ 3g &= 24 \\ g &= 8 \end{aligned}$$

ANSWER: (D)

11. Since N is written out as a sum of powers of 10, then N can be written as $1\,111\,111\,000$, and so the sum of the digits is 7.

ANSWER: (E)

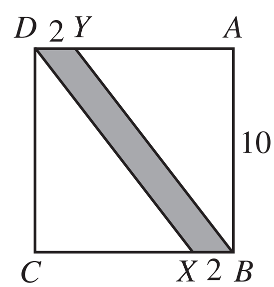
12. *Solution 1*
To get from point B to point C, we go to the left 12 units and up 4. Therefore, for each unit up, we have gone 3 to the left.
To get from B to A, we must go up 1, and since A is on BC, then we must have gone 3 to the left from 9, ie. $a = 9 - 3 = 6$.

Solution 2
Since the three points lie on the same line then

$$\begin{aligned} \text{Slope of } AB &= \text{Slope of } BC \\ \frac{1-0}{a-9} &= \frac{0-4}{9-(-3)} \\ \frac{1}{a-9} &= \frac{-4}{12} \\ 12 &= -4a + 36 \\ 4a &= 24 \\ a &= 6 \end{aligned}$$

ANSWER: (D)

13. *Solution 1*
Since $AY = CX = 8$, then we must have $DY = BX = 2$, and so $DYBX$ is a parallelogram.
If we rotate the picture by 90° clockwise, then we can see that $DYBX$ is a parallelogram with a base of length 2 and a height of 10, ie. it has an area of $bh = 2(10) = 20$.



Solution 2
The area of the shaded region is equal to the area of the entire square minus the areas of the two triangles. Each of the two triangles BAY and DCX is right-angled with one leg of length 10 and the other of length 8.
Therefore,

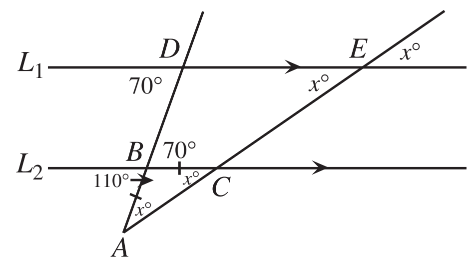
$$\begin{aligned} \text{Area of shaded region} &= \text{Area of square} - \text{Area of triangles} \\ &= (10)^2 - 2\left[\tfrac{1}{2}(8)(10)\right] \\ &= 100 - 2[40] \\ &= 20 \end{aligned}$$

ANSWER: (B)

14. Since the distance covered by Jim in 4 steps is the same as the distance covered by Carly in 3 steps, then the distance covered by Jim in 24 steps is the same as the distance covered by Carly in 18 steps.
Since each of Carly's steps covers 0.5 m, she then covers 9 m in 18 steps, ie. Jim covers 9 m in 24 steps.

ANSWER: (B)

15. We label two more points on the diagram, as shown.
Then $\angle DEC = x^\circ$, since it is equal to its opposite angle.
Since line L_1 is parallel to line L_2, then $\angle DBC = 70^\circ$, since DB is a transversal, and $\angle BCA = x^\circ$ since EC is a transversal.
Since $\angle DBC = 70^\circ$, then $\angle ABC = 110^\circ$, since these angles are supplementary.
Since ΔABC is isosceles, then $\angle BAC = \angle BCA = x^\circ$, and so looking at the sum of the angles in ΔABC, we get



$$
\begin{aligned}
x^\circ + 110^\circ + x^\circ &= 180^\circ \\
2x &= 70 \\
x &= 35
\end{aligned}
$$

ANSWER: (A)

16. Using exponent laws to write all of the factors as product of powers of 2 and 3,

$$\frac{\left(4^{2003}\right)\left(3^{2002}\right)}{\left(6^{2002}\right)\left(2^{2003}\right)} = \frac{\left(\left(2^2\right)^{2003}\right)\left(3^{2002}\right)}{\left((2\cdot 3)^{2002}\right)\left(2^{2003}\right)} = \frac{\left(2^{4006}\right)\left(3^{2002}\right)}{\left(2^{2002}\right)\left(3^{2002}\right)\left(2^{2003}\right)} = \frac{\left(2^{4006}\right)\left(3^{2002}\right)}{\left(2^{4005}\right)\left(3^{2002}\right)} = 2^{4006-4005} = 2^1 = 2$$

ANSWER: (B)

17. Since the largest circle has a radius of 4, its area is $\pi\left(4^2\right) = 16\pi$.
We must calculate the area of each of the shaded regions.
The innermost shaded region is a circle of radius 1, and so it has area $\pi\left(1^2\right) = \pi$.
The outermost shaded region is the region inside a circle of radius 3 and outside a circle of radius 2. Therefore its area is the *difference* between the areas of these two circles, or $\pi\left(3^2\right) - \pi\left(2^2\right) = 5\pi$.
Therefore, the total area of the shaded regions is $\pi + 5\pi = 6\pi$, and the required ratio is $6\pi : 16\pi = 6 : 16 = 3 : 8$.

ANSWER: (E)

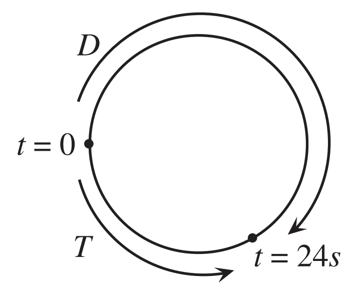
18. Since 496 is less than 2^m, we might think to look for a power of 2 bigger than, but close to 496. $2^9 = 512$ works and in fact $496 = 512 - 16 = 2^9 - 2^4$ and so $m + n = 9 + 4 = 13$.
(This can also be done using an algebraic approach.)

ANSWER: (A)

19. Suppose that the four digit number has digits *abcd*, ie. the product $abcd = 810$.
We must determine how to write 810 as the product of 4 *different* digits, none of which can be 0. So we must start by factoring 810, as $810 = 81 \times 10 = 3^4 \times 2 \times 5$.
So one of the digits must have a factor of 5. But the only non-zero digit having a factor of 5 is 5 itself, so 5 is a digit of the number.
Now we need to find 3 different digits whose product is $3^4 \times 2$.
The only digits with a factor of 3 are 3, 6, and 9, and since we need 4 factors of 3, we must use each of these digits (the 9 contributes 2 factors of 3; the others contribute 1 each). In fact, $3 \times 6 \times 9 = 3^4 \times 2 = 162$.
Therefore, the digits of the number are 3, 5, 6, and 9, and so the sum of the digits is 23.
ANSWER: (C)

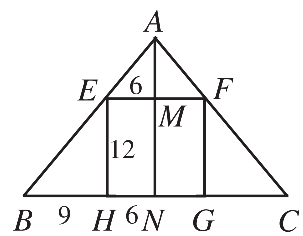
20. The cost to modify the car's engine ($400) is the equivalent of the cost of $\frac{400}{0.80} = 500$ litres of gas. So the car would have to be driven a distance that would save 500 L of gas in order to make up the cost of the modifications.
Originally, the car consumes 8.4 L of gas per 100 km, and after the modifications the car consumes 6.3 L of gas per 100 km, a savings of 2.1 L per 100 km.
Thus, in order to save 500 L of gas, the car would have to be driven $\frac{500}{2.1} \times 100 = 23809.52$ km.
ANSWER: (D)

21. Let's say that time equals 0 seconds when Troye and Daniella first meet. Then at time 24 seconds, they will meet again.
In 24 seconds, how far does Troye get around the track? Since it takes her 56 seconds to complete one lap, then she has made it $\frac{24}{56} = \frac{3}{7}$ of the way around the track.



Since Daniella is running in the opposite direction, then she will go $\frac{4}{7}$ of the way around the track in 24 seconds, and so one complete lap will take her $\frac{7}{4}(24) = 42$ seconds.
ANSWER: (E)

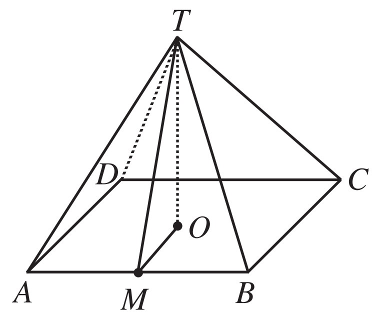
22. Let M be the midpoint of EF and N be the midpoint of HG. By symmetry, N is also the midpoint of BC. Also, the line through A and M will also pass through N, and will be perpendicular to both EF and BC.
Since the side length of the square is 12, then $EM = HN = 6$ and $EH = 12$.
Since we are told that $BC = 30$, then $BN = 15$ and so $BH = 9$.
Since $EFGH$ is a square, then EF is parallel to HG, and so $\angle AEM = \angle EBH$, ie. ΔAME is similar to ΔEHB.
Therefore, $\frac{AM}{6} = \frac{12}{9}$ or $AM = 8$.



Thus, the area of ΔAEF is $\frac{1}{2}(12)(8) = 48\ \text{cm}^2$.

ANSWER: (D)

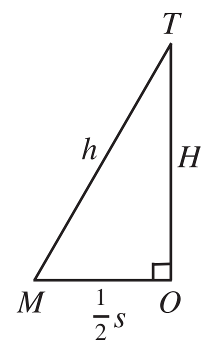
23. We label the pyramid with vertices A, B, C, and D (the square base) and T the "top" vertex. Let M be the midpoint of side AB on the base, and O the centre of the square base.
Since the pyramid has a square base and each of the four triangular faces is identical, then the "top" vertex of the pyramid lies directly above the centre of the base, by symmetry, and so each of the four triangular faces is isosceles.
Join T to O, T to M, and M to O.
Then TO is perpendicular to the square base by the symmetry of the pyramid, and so is perpendicular to OM.
Therefore, triangle TOM is right-angled at O.
Let H be the height of the pyramid, ie. $TO = H$.
Let s be the side length of the base of the pyramid. Then $MO = \frac{1}{2}s$, since O is the centre of the square and M is the midpoint of AB.



Let h be the length of MT. Since M is the midpoint of AB and ΔTAB is isosceles, then TM is perpendicular to AB.
So by Pythagoras, $H^2 = h^2 - \left(\frac{1}{2}s\right)^2 = h^2 - \frac{1}{4}s^2$.
But the base is square, so its area is $s^2 = 1440$, and the area of each of the triangular faces is $\frac{1}{2}sh = 840$, so

$$h^2 = \left(\frac{1680}{s}\right)^2 = \frac{1680^2}{1440} = 1960.$$

Therefore, $H^2 = 1960 - 360 = 1600$, and so $H = 40$.



ANSWER: (B)

24. Since we are looking at choosing four different numbers from the set $\{0, 1, 2, ..., 9\}$, then there is only one way to write them in increasing order. So we only need to look at the number of ways of choosing four numbers so that their sum is a multiple of 3 (that is, we do not need to worry about looking at the order of the choices).
If we take four numbers and add them up, then the fact that the sum is divisible by 3 (or not) is not affected when we subtract a multiple of 3 from any of the four numbers, since the difference between multiples of 3 is a multiple of 3.
Next, we can use the fact that every number can be written as a multiple of 3, or as one more or one less than a multiple of 3, ie. every integer can be written in the form $3n$, $3n+1$ or $3n-1$.
So combining these two facts, we can transform the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into to the collection $\{0, 1, -1, 0, 1, -1, 0, 1, -1, 0\}$, for example by subtracting 6 from 5 to get –1. (A "set" cannot technically have more than one copy of the same element, whereas a "collection" can!)

So now we want to choose 4 numbers from the collection $\{0, 1, -1, 0, 1, -1, 0, 1, -1, 0\}$ whose sum is a multiple of 3 (including possibly 0).
How can we do this?
If we choose 4 zeros, then the sum is $0+0+0+0=0$, which is a multiple of 3.
If we choose 3 zeros, then the remaining number chosen is a 1 or a –1, so the sum is not a multiple of 3.
If we choose 2 zeros, then we can choose two 1's, two –1's, or 1 and –1. Only the third choice gives a multiple of 3.
If we choose 1 zero, then to get a multiple of 3, we must choose three 1's or three –1's. (You might want to check that no other combination works!)
If we choose 0 zeros, then to get a multiple of 3, we must choose two 1's and two –1's (otherwise we choose three of one kind and one of the other, which will not give a multiple of 3).
So now we must count the number of choices for each case:

Case 1: 0, 0, 0, 0

Since there are only four zeros, there is only 1 way to choose them. (Recall that this corresponds to choosing 0, 3, 6 and 9, whose sum is indeed divisible by 3.)

Case 2: 0, 0, 1, –1

We must choose two zeros from four zeros, and one each of 1's and –1's from collections of three.
If we have 4 objects A, B, C, D then the number of ways of choosing 2 objects is 6 (AB, AC, AD, BC, BD, CD), and if we have 3 objects, then the number of ways of choosing 1 object is 3. (This just means that there are 6 ways to choose 2 zeros from 4 possibilities.)
So the total number of choices here is $6 \times 3 \times 3 = 54$, since *for each* choice of two 0's, we have 3 choices for the 1, and 3 choices for the –1.

Case 3: 0, 1, 1, 1

We must choose one 0 from four zeros, and three 1's from 3. There are 4 ways to choose the zero and 1 way to choose the three 1's. Thus there are a total of 4 ways of making this selection.

Case 4: 0, –1, –1, –1

Similarly to Case 3, there are 4 possibilities.

Case 5: 1, 1, –1, –1

We must choose two 1's from three, and two –1's from three. There are 3 ways to make each of these choices, or $3 \times 3 = 9$ ways in total.

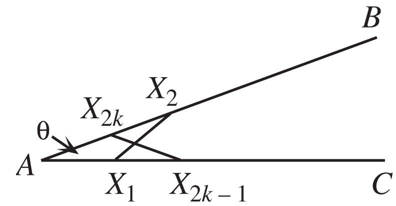
Therefore, there are $1 + 54 + 4 + 4 + 9 = 72$ ways in total of choosing the numbers.

ANSWER: (E)

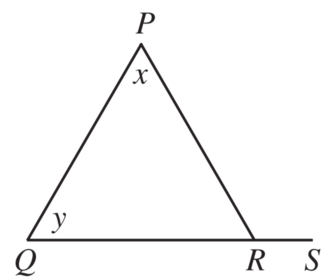
25. Suppose that the angle θ is an acute laceable angle with $2k$ points in the lacing. We need to determine what values of θ are possible.
First, we can note that the diagram must be symmetrical, since

$$AX_1 = X_1X_2 = X_{2k-1}X_{2k} = X_{2k}A$$

and so the two triangles ΔAX_1X_2 and $\Delta AX_{2k}X_{2k-1}$ are isosceles with equal base angles and equal legs, and thus congruent, so $AX_2 = AX_{2k-1}$.



Continuing this, we can show that each pair of corresponding points on the rays AB and AC are the same distance from A.
In particular, $AX_k = AX_{k+1}$. Thus ΔAX_kX_{k+1} is isosceles and so $\angle AX_kX_{k+1} = \angle AX_{k+1}X_k = \frac{1}{2}\left(180^\circ - \theta\right)$.
Next, we will develop a second expression for one of these two angles involving θ.
To do this, we need the fact that if we know two angles of a triangle, then we can calculate the "external angle" of the triangle, ie. in the diagram, $\angle PRS = x + y$, since $\angle PRS = 180^\circ - \angle PRQ = 180^\circ - \left(180^\circ - x - y\right) = x + y$.
Since ΔAX_1X_2 is isosceles, then $\angle X_1AX_2 = \angle AX_2X_1 = \theta$ and so by the external angle, $\angle X_2X_1C = 2\theta$.
Since $\Delta X_1X_2X_3$ is isosceles, then $\angle X_2X_1X_3 = \angle X_2X_3X_1 = 2\theta$, and so by the external angle in ΔAX_2X_3, $\angle X_3X_2C = 3\theta$.



Continuing in this way, $\angle X_3X_2X_4 = \angle X_3X_4X_2 = 3\theta$, and so on, eventually reaching

$$\angle X_kX_{k-1}X_{k+1} = \angle X_kX_{k+1}X_{k-1} = k\theta.$$

(Try this out in the diagram given in the problem.)
Therefore, comparing our two equations for angle $\angle AX_{k+1}X_k = \angle X_{k-1}X_{k+1}X_k$, we obtain

$$\frac{1}{2}\left(180^\circ - \theta\right) = k\theta$$

$$180^\circ = 2k\theta + \theta$$

$$\theta = \frac{180^\circ}{2k+1}$$

Since θ is an integer, $2k+1$ must be a divisor (an odd divisor that is at least 3) of 180.
The divisors of 180 are 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, and 180, with odd divisors 1, 3, 5, 9, 15, and 45.
Therefore, ignoring the 1, there are 5 possibilities for θ to be a laceable acute angle, namely 60°, 36°, 20°, 12°, and 4°.

ANSWER: (C)