Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2001 Solutions
Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

Part A

1. The value of $\frac{5(6)-3(4)}{6+3}$ is

(**A**) 1 (**B**) 2 (**C**) 6 (**D**) 12 (**E**) 31

Solution
By evaluating the numerator and denominator we have,

$$\frac{5(6)-3(4)}{6+3}=\frac{30-12}{9}=\frac{18}{9}=2.$$

ANSWER: (B)

2. When $\frac{1}{4}$ of 15 is multiplied by $\frac{1}{3}$ of 10, the answer is

(**A**) 5 (**B**) $\frac{25}{2}$ (**C**) $\frac{85}{12}$ (**D**) $\frac{99}{8}$ (**E**) $\frac{25}{7}$

Solution
Evaluating gives,

$$\left[\tfrac{1}{4}(15)\right]\left[\tfrac{1}{3}(10)\right]$$
$$=\frac{1}{\cancel{4}_2}(\cancel{15}^{5})\frac{1}{\cancel{3}_1}(\cancel{10}^{5})$$
$$=\frac{25}{2}$$

ANSWER: (B)

3. If $x=\frac{1}{4}$, which of the following has the largest value?

(**A**) x (**B**) x^2 (**C**) $\frac{1}{2}x$ (**D**) $\frac{1}{x}$ (**E**) $\sqrt{x}$

Solution
If we calculate the value of the given expressions, we get

(**A**) $\frac{1}{4}$

(**B**) $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

(**C**) $\frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$

(**D**) $\frac{1}{\frac{1}{4}} = 1\times 4 = 4$

(**E**) $\sqrt{\frac{1}{4}} = \frac{1}{2}$

ANSWER: (D)

4. In a school, 30 boys and 20 girls entered the Cayley competition. Certificates were awarded to 10% of the boys and 20% of the girls. Of the students who participated, the percentage that received

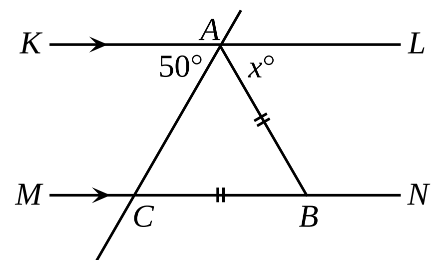
certificates was

(**A**) 14 (**B**) 15 (**C**) 16 (**D**) 30 (**E**) 50

Solution

If 30 boys entered the Cayley competition and 10% of them received certificates, this implies that $(0.1)(30)$ or 3 boys received certificates. Of the 20 girls who entered the competition $(0.2)(20)$ or 4 girls received certificates. This implies that 7 students in total out of 50 received certificates.
Thus 14% of the students in total received certificates. ANSWER: (A)

5. In the diagram, KL is parallel to MN, $AB = BC$, and $\angle KAC = 50°$. The value of x is



(**A**) 40 (**B**) 65 (**C**) 25
(**D**) 100 (**E**) 80

Solution

Since KL is parallel to MN, $\angle ACB$ and $\angle KAC$ are alternate angles. Thus, $\angle ACB = 50°$.
We are given that ΔBCA is isosceles, so $\angle BCA = \angle BAC = 50°$. We know that since angles KAC, BAC and LAB have a sum of $180°$,

$$50° + 50° + x° = 180°$$
$$x = 80.$$

ANSWER: (E)

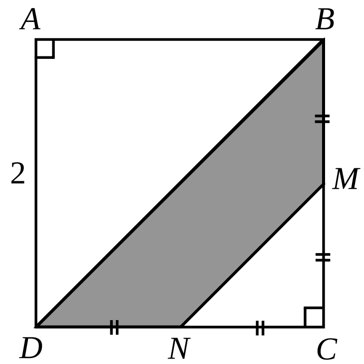
6. Dean scored a total of 252 points in 28 basketball games. Ruth played 10 fewer games than Dean. Her scoring average was 0.5 points per game higher than Dean's scoring average. How many points, in total, did Ruth score?

(**A**) 153 (**B**) 171 (**C**) 180 (**D**) 266 (**E**) 144

Solution

If Dean scored 252 points in 28 games this implies that he averages $\frac{252}{28}$ or 9 points per game.
Ruth must then have averaged 9.5 points in each of the 18 games she played. In total she scored 9.5×18 or 171 points. ANSWER: (B)

7. In the diagram, square $ABCD$ has side length 2, with M the midpoint of BC and N the midpoint of CD. The area of the shaded region $BMND$ is



(**A**) 1 (**B**) $2\sqrt{2}$ (**C**) $\frac{4}{3}$

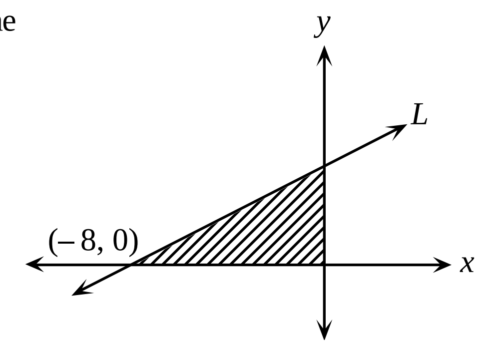
(**D**) $\frac{3}{2}$ (**E**) $4-\frac{3}{2}\sqrt{2}$

Solution

The area of ΔMNC is $\frac{1}{2}(1)(1)=\frac{1}{2}$. Since ΔBDC is half the square, it will have an area of 2. Since the shaded region has an area equal to that of ΔBDC minus the area of ΔMNC, its area will be $2-\frac{1}{2}=\frac{3}{2}$.

ANSWER: (D)

8. The line L crosses the x-axis at $(-8, 0)$. The area of the shaded region is 16. What is the slope of the line L?



(**A**) $\frac{1}{2}$ (**B**) 4 (**C**) $-\frac{1}{2}$

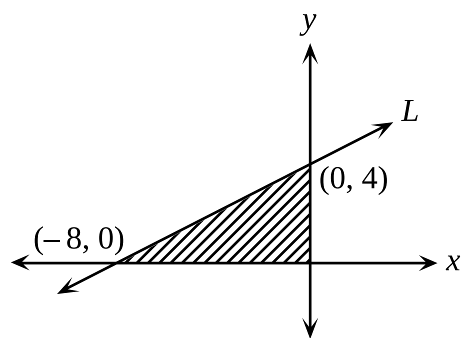
(**D**) 2 (**E**) -2

Solution

If the area of the shaded region is 16 and its base has a length of 8, its height must then be 4.

Thus we have the changes noted in the diagram.

Thus the slope is $\dfrac{4-0}{0-(-8)}=\dfrac{1}{2}$ or $\dfrac{1}{2}$ because the line slopes down from right to left and the line has a rise of 4 and a run of 8.



Area $=\frac{1}{2}|-8||4|=16$

ANSWER: (A)

9. If $\left[\left(10^3\right)\left(10^x\right)\right]^2=10^{18}$, the value of x is

(**A**) $\sqrt{2}$ (**B**) 12 (**C**) 6 (**D**) 1 (**E**) 3

Solution
If we simplify using laws of exponents, we will have,

$$\left(10^{3+x}\right)^2 = 10^{18}$$
$$10^{6+2x} = 10^{18}.$$

In order that the left and right side be equal, it is necessary that exponents be equal.
Thus, $6 + 2x = 18$
$$2x = 12$$
$$x = 6.$$

ANSWER: (C)

10. The sum of five consecutive integers is 75. The sum of the largest and smallest of these five integers is

(**A**) 15 (**B**) 25 (**C**) 26 (**D**) 30 (**E**) 32

Solution
There are a variety of ways of approaching this problem. The easiest is to represent the integers as $x-2$, $x-1$, x, $x+1$, and $x+2$.
Thus, $(x-2)+(x-1)+x+(x+1)+(x+2) = 75$
$$5x = 75$$
$$x = 15.$$
The five consecutive integers are 13, 14, 15, 16, and 17.
The required sum is $13+17=30$. ANSWER: (D)

Part B

11. When a positive integer N is divided by 60, the remainder is 49. When N is divided by 15, the remainder is

(**A**) 0 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 8

Solution
This problem can be done in a number of ways. The easiest way is to consider that if N is divided by 60 to achieve a remainder of 49, it must be a number of the form, $60k+49$, $k = 0, 1, 2, \ldots$.
This implies that the smallest number to meet the requirements is 49 itself. If we divide 49 by 15 we get a remainder of 4. Or, if $k=1$ in our formula then the next number to satisfy the requirements is 109 which when divided by 15 gives 4 as the remainder. ANSWER: (C)

12. The 6 members of an executive committee want to call a meeting. Each of them phones 6 different

people, who in turn each calls 6 other people. If no one is called more than once, how many people will know about the meeting?

(**A**) 18 (**B**) 36 (**C**) 216 (**D**) 252 (**E**) 258

Solution
If 6 people each call 6 other people in the first round of calls, there will be 36 people making 6 calls each for an additional 216 calls. Altogether, there will be the original 6, followed by 36 who in turn phone another 216.
In total, there are $6+36+216=258$. ANSWER: (E)

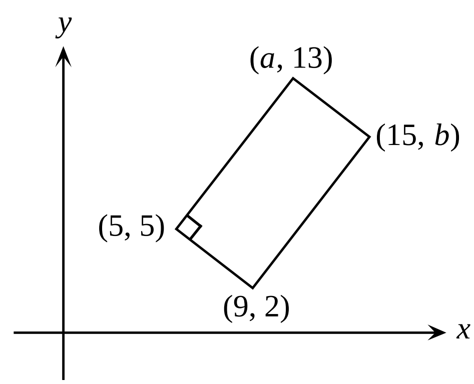
13. The sequences 3, 20, 37, 54, 71, … and 16, 27, 38, 49, 60, 71, … each have 71 as a common term. The next term that these sequences have in common is

(**A**) 115 (**B**) 187 (**C**) 258 (**D**) 445 (**E**) 1006

Solution
The first sequence increases by a constant value of 17 and the second by a constant value of 11.
After 71, the next common term will be 71 plus the Lowest Common Multiple of 11 and 17. Since the L.C.M. of 11 and 17 is 187 the next term will be $71+187=258$. ANSWER: (C)

14. In the rectangle shown, the value of $a-b$ is

(**A**) -3 (**B**) -1 (**C**) 0
(**D**) 3 (**E**) 1



Solution
To go from the point $(5,5)$ to the point $(9,2)$ we must move over 4 and down 3.
Since we are dealing with a rectangle, the same must be true for $(a,13)$ and $(15,b)$.
Thus, $a+4=15$ and $13-3=b$. From this, $a=11$ and $b=10$. So $a-b=11-10=1$.
ANSWER: (E)

15. A small island has $\frac{2}{5}$ of its surface covered by forest and $\frac{1}{4}$ of the remainder of its surface by sand dunes. The island also has 90 hectares covered by farm land. If the island is made up of only forest, sand dunes and farm land, what is the total area of the island, to the nearest hectare?

(**A**) 163 (**B**) 120 (**C**) 200 (**D**) 138 (**E**) 257

Solution

If $\frac{2}{5}$ of an island is covered by forest then $\frac{3}{5}$ of the island is made up of sand dunes and farm land. Since $\frac{1}{4}\times\frac{3}{5}=\frac{3}{20}$ is made up of sand dunes this implies that $\frac{2}{5}+\frac{3}{20}=\frac{11}{20}$ of the island is made up of forest and sand dunes. Thus $\frac{9}{20}$ of the island, or 90 hectares, is made up of farm land.

Thus, the island must be $\frac{20}{\cancel{9}_1}=\cancel{90}^{10}$ or 200 hectares in total. ANSWER: (C)

16. How many integer values of x satisfy $\frac{x-1}{3}<\frac{5}{7}<\frac{x+4}{5}$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

Solution

If we multiply all three fractions by $3(5)(7)$ we have,

$$(\cancel{3})(5)(7)\frac{(x-1)}{\cancel{3}}<(3)(5)(\cancel{7})\frac{5}{\cancel{7}}<(3)(\cancel{5})(7)\frac{x+4}{\cancel{5}}$$

$$35(x-1)<75<21(x+4)$$

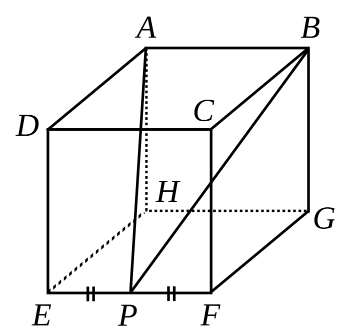
In order to satisfy this inequality then,

$$\begin{array}{rcr} 35(x-1)<75 & \text{and} & 21(x+4)>75 \\ 35x-35<75 & \text{and} & 21x+84>75 \\ 35x<110 & & 21x>-9 \\ x<3\frac{1}{7} & & x>-\frac{9}{21} \end{array}$$

The only integers to satisfy both conditions are then in the set $\{0, 1, 2, 3\}$. ANSWER: (E)

17. $ABCDEFGH$ is a cube having a side length of 2. P is the midpoint of EF, as shown. The area of ΔAPB is

(**A**) $\sqrt{8}$ (**B**) 3 (**C**) 6
(**D**) $\sqrt{2}$ (**E**) $\sqrt{32}$



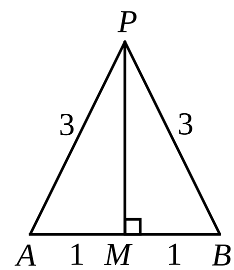
Solution

By symmetry, the lengths of AP and BP will be equal, and $AP = \sqrt{AD^2 + DE^2 + EP^2} = \sqrt{2^2 + 2^2 + 1^2} = 3$.

If M is the midpoint of AB, then PM is perpendicular to AB. By Pythagoras, $MP = \sqrt{3^2 - 1^2} = \sqrt{8}$.

So the area of ΔAPB is

$$\text{Area} = \tfrac{1}{2}(2)\left(\sqrt{8}\right) = \sqrt{8}.$$



ANSWER: (A)

18. How many five-digit positive integers, divisible by 9, can be written using only the digits 3 and 6?

(**A**) 5 (**B**) 2 (**C**) 12 (**D**) 10 (**E**) 8

Solution

If a five-digit number is composed of only 3's and 6's, there are just two cases to consider because these are the only ways that the digital sum can be a multiple of 9.

Case 1 one 6, four 3's

In this case, there are five numbers: $63\,333$, $36\,333$, $33\,633$, $33\,363$ and $33\,336$.

Case 2 one 3, four 6's

In this case there are five numbers: $36\,666$, $63\,666$, $66\,366$, $66\,636$ and $66\,663$.

In total there are 10 possible numbers. ANSWER: (D)

19. Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is

(**A**) 34 (**B**) 35 (**C**) 36 (**D**) 37 (**E**) 38

Solution

Let the three numbers be a, b and c.

We construct the first equation to be,

$$a + \frac{b+c}{2} = 65.$$

Or, $2a + b + c = 130$.

Similarly we construct the two other equations to be,

$$a + 2b + c = 138$$

and $$a + b + 2c = 152.$$

If we add the three equations we obtain,

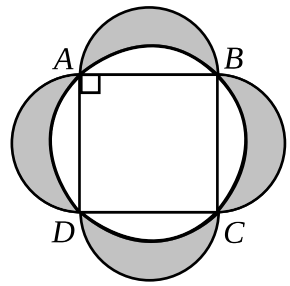
$$4a + 4b + 4c = 420.$$

The average is $\frac{4(a+b+c)}{12} = \frac{420}{12}$.

Or, $\frac{a+b+c}{3} = 35$.

ANSWER: (B)

20. Square $ABCD$ with side length 2 is inscribed in a circle, as shown. Using each side of the square as a diameter, semi-circular arcs are drawn. The area of the shaded region outside the circle and inside the semi-circles is



(A) π **(B)** 4 **(C)** $2\pi - 2$
(D) $\pi + 1$ **(E)** $2\pi - 4$

Solution

The side length of the square is 2 and thus the diameter of the circle has length $2\sqrt{2}$ which is also the length of the diagonal AC (or BD). The area of the circle is thus $\pi\left(\sqrt{2}\right)^2 = 2\pi$. Since the side length of the square is 2, it will have an area of 4. From this, we calculate the area of the circle outside the square to be $2\pi - 4$. To calculate the shaded area, we first calculate the area of each semi-circle. Each of the semi circles has a radius of 1 meaning that each semi-circle will have an area of $\frac{1}{2}\left[\pi(1)^2\right] = \frac{1}{2}\pi$. In total, the four semi-circles have an area of 2π. Thus the shaded area has an area of $2\pi - (2\pi - 4) = 4$.

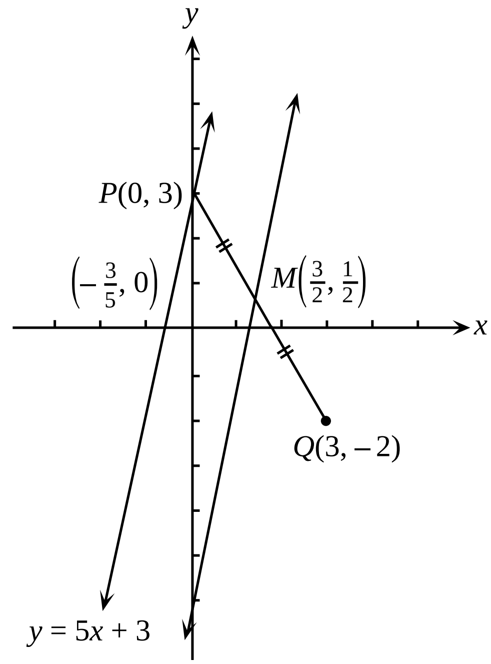
ANSWER: (B)

Part C

21. Point P is on the line $y = 5x + 3$. The coordinates of point Q are $(3, -2)$. If M is the midpoint of PQ, then M must lie on the line

(A) $y = \frac{5}{2}x - \frac{7}{2}$ **(B)** $y = 5x + 1$ **(C)** $y = -\frac{1}{5}x - \frac{7}{5}$ **(D)** $y = \frac{5}{2}x + \frac{1}{2}$ **(E)** $y = 5x - 7$

We start by drawing a diagram and labelling the intercepts.



Solution 1

Since the point P is on the line $y = 5x + 3$, select $P(0, 3)$ as a point on this line.

The midpoint of PQ is $M\left(\frac{3+0}{2}, \frac{-2+3}{2}\right) = M\left(\frac{3}{2}, \frac{1}{2}\right)$.

The required line must contain M and be midway between the given point and $y = 5x + 3$. The only possible line meeting this requirement is the line containing $M\left(\frac{3}{2}, \frac{1}{2}\right)$ and which has a slope of 5. The required line will this have as its equation

$$y - \tfrac{1}{2} = 5\left(x - \tfrac{3}{2}\right)$$

or, $$y = 5x - 7.$$

Solution 2

Let a general point on the line $y = 5x + 3$ be represented by $(a, 5a + 3)$. Also, let a point on the required line be $M(x, y)$. Since $M(x, y)$ is the midpoint of PQ then

(1) $x = \dfrac{a+3}{2}$ and (2) $y = \dfrac{(5a+3)+(-2)}{2}$

$$y = \frac{5a+1}{2}$$

Solving (1) for a, we have $a = 2x - 3$ and solving (2) for a, we have $\dfrac{2y-1}{5} = a$.

Equating gives, $2x - 3 = \dfrac{2y-1}{5}$

$$10x - 15 = 2y - 1$$

or, $$y = 5x - 7.$$

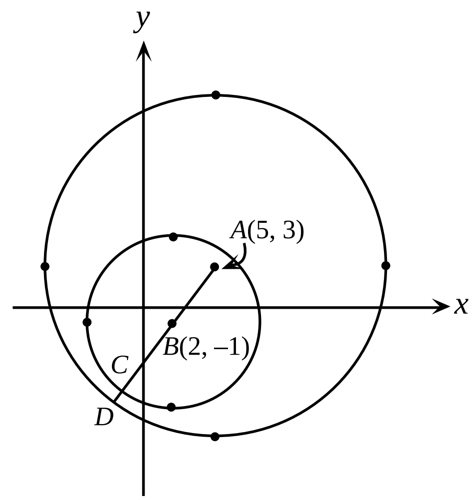
ANSWER: (E)

22. What is the shortest distance between two circles, the first having centre $A(5, 3)$ and radius 12, and the other with centre $B(2, -1)$ and radius 6?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

We start by drawing the two circles where the larger circle has centre $A(5,3)$ and the smaller circle has centre $B(2,-1)$. A line is drawn from A, through B to meet the circumference of the smaller circle at C and the circumference of the larger circle at D. The length CD is the desired length. The length from A to D is given to be 12 and the length from B to C is 6. We calculate the length of AB to be, $\sqrt{[3-(-1)]^2+(5-2)^2}=\sqrt{16+9}=5$. To find CD, we calculate as follows,



$$\begin{aligned} CD &= AD-(AB+BC) \\ &= 12-(5+6) \\ &= 1. \end{aligned}$$

ANSWER: (A)

23. A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?

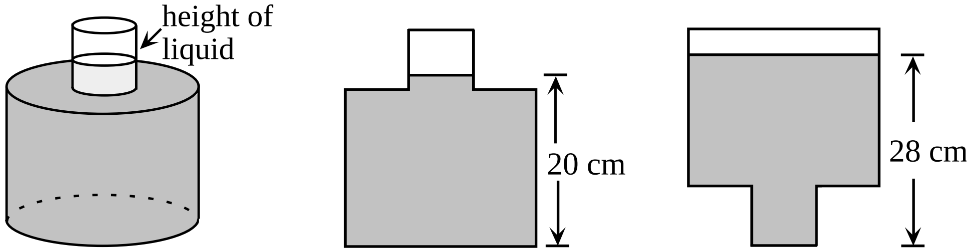


Figure A Figure B Figure C

(**A**) 29 (**B**) 30 (**C**) 31 (**D**) 32 (**E**) 48

Solution

We'll start by representing the height of the large cylinder as h_1 and the height of the small cylinder as h_2. For simplicity, we'll let $x = h_1 + h_2$.

If the bottom cylinder is completely filled and the top cylinder is only partially filled the top cylinder will have a cylindrical space that is not filled. This cylindrical space will have a height equal to $x-20$ and a volume equal to, $\pi(1)^2(x-20)$.

Similarly, if we turn the cylinder upside down there will be a cylindrical space unfilled that will have a height equal to $x-28$ and a volume equal to, $\pi(3)^2(x-28)$.

Since these two unoccupied spaces must be equal, we then have,

$$\begin{aligned}\pi(1)^2(x-20) &= \pi(3)^2(x-28)\\ x-20 &= 9x-252\\ 8x &= 232\\ x &= 29.\end{aligned}$$

Therefore, the total height is 29. ANSWER: (A)

24. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2882 is a four-digit palindrome and 49194 is a five-digit palindrome. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?

(**A**) 28 (**B**) 32 (**C**) 36 (**D**) 40 (**E**) 44

Solution

Our first observation is that since we are adding two four-digit palindromes to form a five-digit palindrome then the following must be true,

$$\begin{array}{r} a\,b\,b\,a\\ \underline{c\,d\,d\,c}\,.\\ 1\,e\,f\,e\,1\end{array}$$

(i.e. the first digit of the 5-digit palindrome is 1.)

From this, we can see that $a+c=11$ since $a+c$ has a units digit of 1 and $10<a+c<20$.

We first note that there are four possibilities for *a* and *c*. We list the possibilities:

a	2	3	4	5
c	9	8	7	6

Note that there are only four possibilities here.

(If we extended this table, we would get an additional four possibilities which would be duplicates of these four, with *a* and *c* reversed.)

Let us consider one case, say $a=2$ and $c=9$.

$$\begin{array}{r} 2\,b\,b\,2\\ \underline{9\,d\,d\,9}\\ 1\,e\,f\,e\,1\end{array}$$

From this, we can only get palindromes in two ways. To see this we note that *e* is either 1 or 2 depending on whether we get a carry from the previous column (we see this looking at the thousands digit of 1 *e f e* 1). If $e=1$, then $b+d$ has no carry and so looking at the tens digit $e=1$, we see that $b+d=0$ to get this digit.

If $e = 2$, we do get a carry from $b + d$, so looking again at the tens digit $e = 2$, we see that $b + d = 11$.

Possibility 1 $b = d = 0$

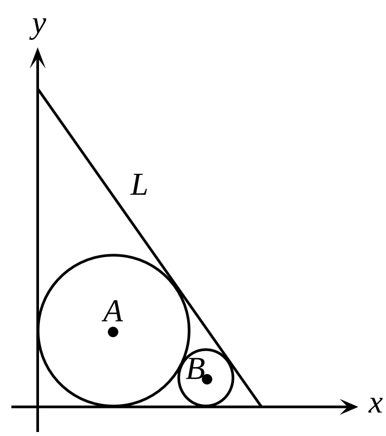
Since there are only four possibilities for *a* and *c* and just one way of selecting *b* and *d* so that $b + d = 0$ for each possibility, there are just four possibilities.

Possibility 2 $b + d = 11$

For each of the four possible ways of choosing *a* and *c*, there are eight ways of choosing *b* and *d* so that $b + d = 11$ thus giving 32 possibilities.
This gives a total of $4 + 32 = 36$ possibilities.

ANSWER: (C)

25. The circle with centre *A* has radius 3 and is tangent to both the positive *x*-axis and positive *y*-axis, as shown. Also, the circle with centre *B* has radius 1 and is tangent to both the positive *x*-axis and the circle with centre *A*. The line *L* is tangent to both circles. The *y*-intercept of line *L* is

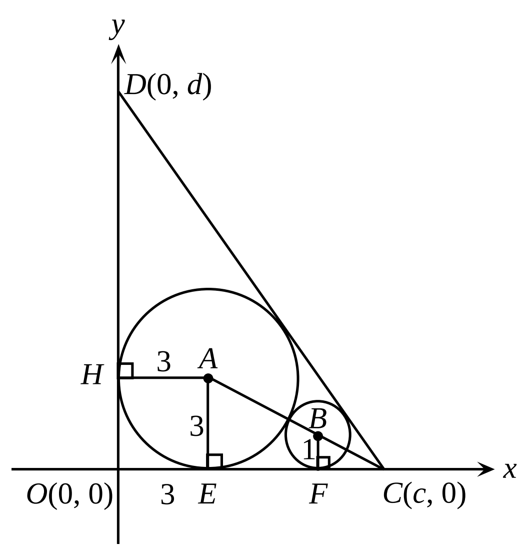


(**A**) $3 + 6\sqrt{3}$ (**B**) $10 + 3\sqrt{2}$ (**C**) $8\sqrt{3}$

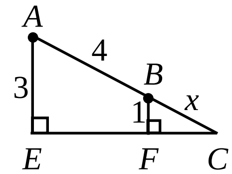
(**D**) $10 + 2\sqrt{3}$ (**E**) $9 + 3\sqrt{3}$

Solution

We start by drawing a line from point *C* that will pass through *A* and *B*. From *A* and *B*, we drop perpendiculars to the points of tangency on the *x*-axis and label these points as *E* and *F* as shown. We also drop a perpendicular from *A* to the *y*-axis which makes $AH = AE = 3$.



Extracting ΔCAE from the diagram and labelling with the given information we would have the following noted in the diagram.



If we represent the distance from C to B as x and recognize that ΔCBF is similar to ΔCAE,

$$\frac{x}{1} = \frac{x+4}{3}$$
$$x = 2.$$

In ΔCBF, $FC^2 = 2^2 - 1^2 = 3$

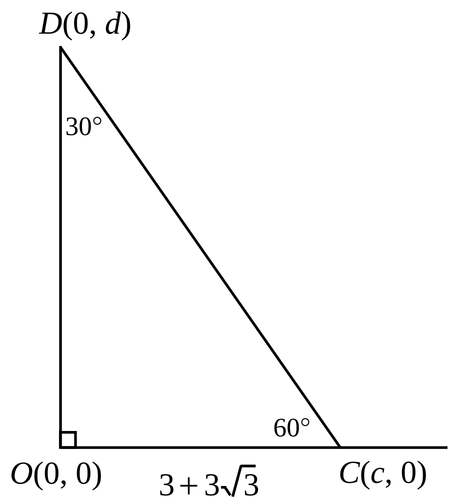
$$FC = \sqrt{3},\ (FC > 0).$$

This implies that $\angle BCF = 30°$ and $\angle OCD = 60°$. Therefore $EF = 2\sqrt{3}$, from similar triangles again.

This now gives us the diagram shown.

Thus, $d = \sqrt{3}\left(3 + 3\sqrt{3}\right)$

$$= 3\sqrt{3} + 9.$$



ANSWER: (E)

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