Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for the

NATIONAL BANK OF CANADA

Awards

Tuesday, April 21, 1998

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts

1. **SHORT ANSWER** parts are indicated like this: 
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts

1. **FULL SOLUTION** parts are indicated like this: 
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Place all answers in the answer booklet provided.
3. For questions marked "  ", full marks will be given for a correct answer which is placed in the box in the answer booklet. Part marks will be awarded ***only if relevant work*** is shown in the space provided in the answer booklet.
4. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1.  (a) If one root of $x^2+2x-c=0$ is $x=1$, what is the value of c?

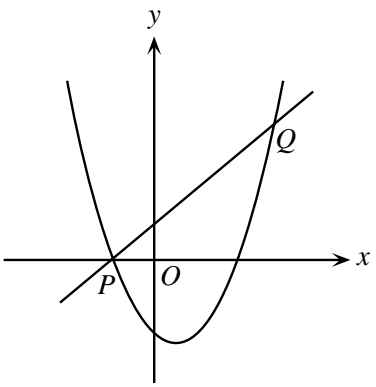
 (b) If $2^{2x-4}=8$, what is the value of x?

 (c) Two perpendicular lines with x-intercepts -2 and 8 intersect at $(0, b)$. Determine all values of b.

2.  (a) The vertex of $y=(x-1)^2+b$ has coordinates $(1, 3)$. What is the y-intercept of this parabola?

 (b) What is the area of ΔABC with vertices $A(-3, 1)$, $B(5, 1)$ and $C(8, 7)$?

 (c) In the diagram, the line $y=x+1$ intersects the parabola $y=x^2-3x-4$ at the points P and Q. Determine the coordinates of P and Q.



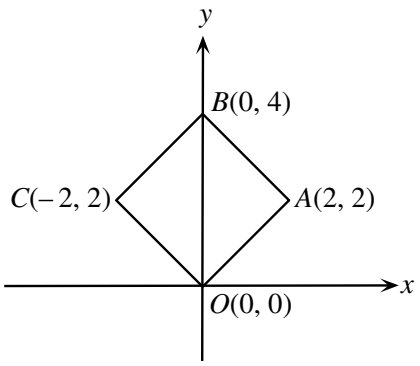
3.  (a) The graph of $y=m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

 (b) Jane bought 100 shares of stock at \$10.00 per share. When the shares increased to a value of \$$N$ each, she made a charitable donation of all the shares to the Euclid Foundation. She received a tax refund of 60% on the total value of her donation. However, she had to pay a tax of 20% on the increase in the value of the stock. Determine the value of N if the difference between her tax refund and the tax paid was \$1000.

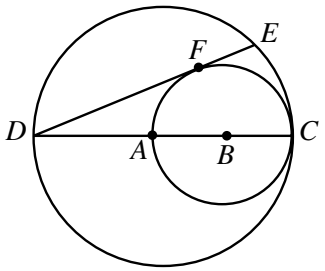
4.  (a) Consider the sequence $t_1=1$, $t_2=-1$ and $t_n=\left(\frac{n-3}{n-1}\right)t_{n-2}$ where $n\geq 3$. What is the value of t_{1998}?

 (b) The nth term of an arithmetic sequence is given by $t_n=555-7n$.
If $S_n=t_1+t_2+...+t_n$, determine the smallest value of n for which $S_n<0$.

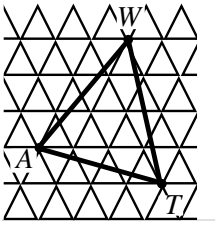
5. (a) A square $OABC$ is drawn with vertices as shown. Find the equation of the circle with largest area that can be drawn inside the square.



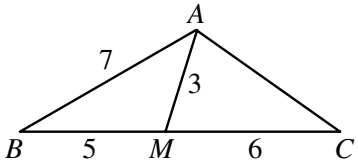
(b) In the diagram, DC is a diameter of the larger circle centred at A, and AC is a diameter of the smaller circle centred at B. If DE is tangent to the smaller circle at F, and $DC = 12$, determine the length of DE.



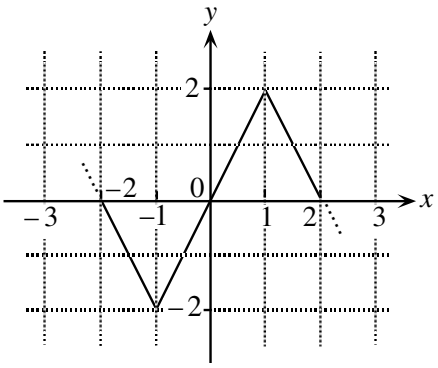
6. (a) In the grid, each small equilateral triangle has side length 1. If the vertices of ΔWAT are themselves vertices of small equilateral triangles, what is the area of ΔWAT?



(b) In ΔABC, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



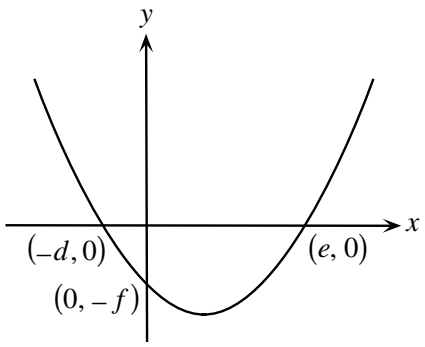
7. (a) The function $f(x)$ has period 4. The graph of one period of $y = f(x)$ is shown in the diagram. Sketch the graph of $y = \frac{1}{2}[f(x-1) + f(x+3)]$, for $-2 \le x \le 2$.



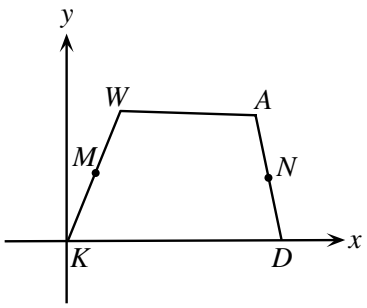
(b) If x and y are real numbers, determine all solutions (x, y) of the system of equations

$$x^2 - xy + 8 = 0$$
$$x^2 - 8x + y = 0.$$

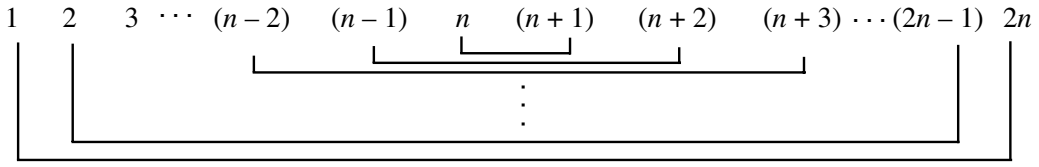
8. (a) In the graph, the parabola $y = x^2$ has been translated to the position shown.
Prove that $de = f$.



(b) In quadrilateral $KWAD$, the midpoints of KW and AD are M and N respectively. If $MN = \frac{1}{2}(AW + DK)$, prove that WA is parallel to KD.



9. Consider the first $2n$ natural numbers. Pair off the numbers, as shown, and multiply the two members of each pair. Prove that there is no value of n for which two of the n products are equal.



10. The equations $x^2 + 5x + 6 = 0$ and $x^2 + 5x - 6 = 0$ ***each*** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.

(a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ ***both*** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$. (i.e. (a, b, p) is a Pythagorean triple).

(b) Determine q in terms of a and b.