35th Anniversary
1963 – 1998

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1998 Solutions
Cayley Contest (Grade 10)

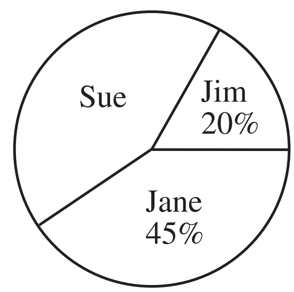
for the
NATIONAL BANK OF CANADA
Awards

PART A:

1. The value of $(0.3)^2 + 0.1$ is
(**A**) 0.7 (**B**) 1 (**C**) 0.1 (**D**) 0.19 (**E**) 0.109

Solution
$$(0.3)^2 + 0.1 = 0.09 + 0.1$$
$$= 0.19$$
ANSWER: (D)

2. The pie chart shows a percentage breakdown of 1000 votes in a student election. How many votes did Sue receive?
(**A**) 550 (**B**) 350 (**C**) 330
(**D**) 450 (**E**) 935



Solution
Sue received $100 - (20 + 45) = 35$ percent of the total number of votes. Since there was a total of 1000 votes, Sue received $0.35(1000) = 350$ votes.
ANSWER: (B)

3. The expression $\frac{a^9 \times a^{15}}{a^3}$ is equal to
(**A**) a^{45} (**B**) a^8 (**C**) a^{18} (**D**) a^{14} (**E**) a^{21}

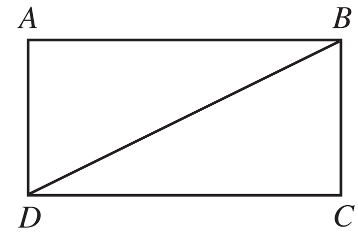
Solution
$$\frac{a^9 \times a^{15}}{a^3} = \frac{a^{24}}{a^3}$$
$$= a^{21}$$
ANSWER: (E)

4. The product of two positive integers p and q is 100. What is the largest possible value of $p + q$?
(**A**) 52 (**B**) 101 (**C**) 20 (**D**) 29 (**E**) 25

Solution
The pairs of positive integers whose product is 100 are: 1 and 100, 2 and 50, 4 and 25, 5 and 20, 10 and 10. The pair with the largest sum is 1 and 100. The sum is 101.
ANSWER: (B)

5. In the diagram, $ABCD$ is a rectangle with $DC = 12$. If the area of triangle BDC is 30, what is the perimeter of rectangle $ABCD$?

(**A**) 34 (**B**) 44 (**C**) 30
(**D**) 29 (**E**) 60



Solution

Since the area of ΔBDC is 30, we know

$$\frac{1}{2}(12)(BC) = 30$$
$$6(BC) = 30$$
$$BC = 5$$

Thus, the perimeter of rectangle $ABCD$ is $2(12) + 2(5) = 34$. ANSWER: (A)

6. If $x = 2$ is a solution of the equation $qx - 3 = 11$, the value of q is

(**A**) 4 (**B**) 7 (**C**) 14 (**D**) -7 (**E**) -4

Solution

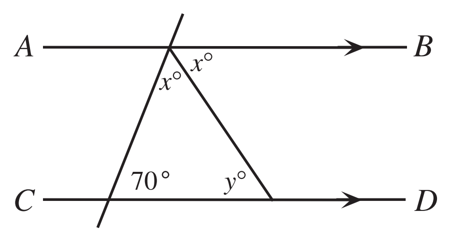
If $x = 2$ is a solution of $qx - 3 = 11$, then

$$q(2) - 3 = 11$$
$$2q = 14$$
$$q = 7$$

ANSWER: (B)

7. In the diagram, AB is parallel to CD. What is the value of y?

(**A**) 75 (**B**) 40 (**C**) 35
(**D**) 55 (**E**) 50

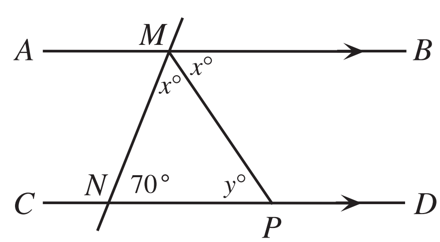


Solution

Since AB is paralled to CD, then $\angle BMN + \angle MND = 180$.

Thus, $2x + 70 = 180$
$$2x = 110$$
$$x = 55$$

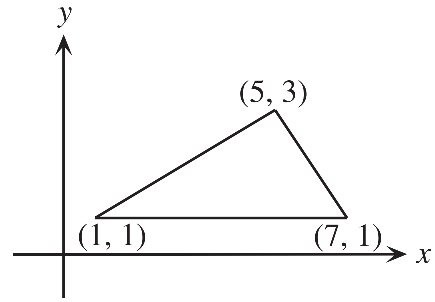
Using ΔMNP, $y = 180 - (70 + 55)$
$$= 55$$



ANSWER: (D)

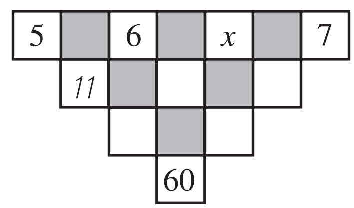
8. The vertices of a triangle have coordinates $(1,1)$, $(7,1)$ and $(5,3)$. What is the area of this triangle?
(**A**) 12 (**B**) 8 (**C**) 6 (**D**) 7 (**E**) 9

Solution
Draw the triangle on the coordinate axes. This triangle has a base of 6 and a height of 2. Its area is $\frac{1}{2}(6)(2) = 6$.



ANSWER: (C)

9. The number in an unshaded square is obtained by adding the numbers connected to it from the row above. (The '11' is one such number.) The value of x must be
(**A**) 4 (**B**) 6 (**C**) 9
(**D**) 15 (**E**) 10



Solution
The three entries in row two, from left to right, are 11, $6+x$, and $x+7$. The two entries in row three, from left to right, are $11+(6+x)=17+x$ and $(6+x)+(x+7)=2x+13$. The single entry in row four is $(17+x)+(2x+13)=3x+30$.

Thus, $3x+30=60$
$3x=30$
$x=10$

ANSWER: (E)

10. The sum of the digits of a five-digit positive integer is 2. (A five-digit integer cannot start with zero.) The number of such integers is
(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution
If the sum of the digits of a five-digit positive integer is 2, then the only possible integers are 20 000, 11 000, 10 100, 10 010, and 10 001. There are 5 such integers. ANSWER: (E)

PART B:

11. If $x+y+z=25$, $x+y=19$ and $y+z=18$, then y equals
(**A**) 13 (**B**) 17 (**C**) 12 (**D**) 6 (**E**) -6

Solution

We are given that $x + y + z = 25$ (1)

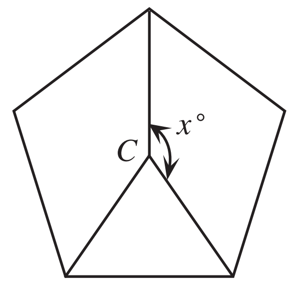
$$x + y = 19 \quad (2)$$

$$y + z = 18 \quad (3)$$

Add equations (2) and (3) to get $x + 2y + z = 37$ (4). Subtract equation (1) from equation (4) to get $y = 12$.

ANSWER: (C)

12. A regular pentagon with centre C is shown. The value of x is

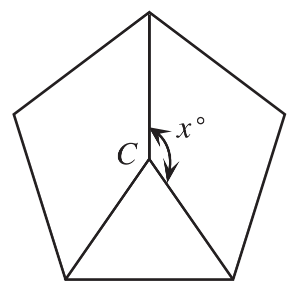


(**A**) 144 (**B**) 150 (**C**) 120
(**D**) 108 (**E**) 72

Solution

Join C to each of the remaining vertices, as shown. Since the pentagon is regular, each of the small angles at C has measure $360^\circ \div 5 = 72^\circ$.

Thus, the value of x is $2(72) = 144$.



ANSWER: (A)

13. If the surface area of a cube is 54, what is its volume?

(**A**) 36 (**B**) 9 (**C**) $\frac{81\sqrt{3}}{8}$ (**D**) 27 (**E**) $162\sqrt{6}$

Solution

Let each edge of the cube have length x. Then each face of the cube has area x^2. Since a cube has six faces, the surface area of the cube is $6x^2$.

We know $6x^2 = 54$

$$x^2 = 9$$

$$x = 3$$

Since each edge of the cube has length 3, the volume of the cube is $3^3 = 27$.

ANSWER: (D)

14. The number of solutions (x, y) of the equation $3x + y = 100$, where x and y are positive integers, is

(**A**) 33 (**B**) 35 (**C**) 100 (**D**) 101 (**E**) 97

Solution

Rewrite the given equation as $x = \frac{100 - y}{3}$. Since x must be a positive integer, $100 - y$ must be divisible by 3. Since y must also be a positive integer, the only possible values of y are 1, 4, 7, 10, 13, ... , 94, and 97. Thus, there are 33 possible values for y, and 33 solutions (x, y) that meet the given conditions.

ANSWER: (A)

15. If $\sqrt{y-5} = 5$ and $2^x = 8$, then $x + y$ equals

(**A**) 13 (**B**) 28 (**C**) 33 (**D**) 35 (**E**) 38

Solution

Since $\sqrt{y-5} = 5$, then

$$\left(\sqrt{y-5}\right)^2 = 5^2$$
$$y - 5 = 25$$
$$y = 30$$

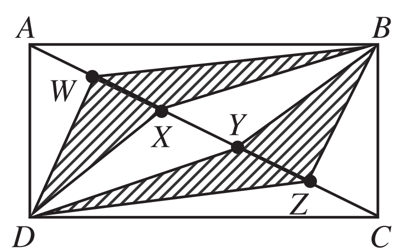
Also, since $2^x = 8$, then

$$2^x = 2^3$$
$$x = 3$$

Thus, $x + y = 33$.

ANSWER: (C)

16. Rectangle $ABCD$ has length 9 and width 5. Diagonal AC is divided into 5 equal parts at W, X, Y, and Z. Determine the area of the shaded region.



(**A**) 36 (**B**) $\frac{36}{5}$ (**C**) 18

(**D**) $\frac{4\sqrt{106}}{5}$ (**E**) $\frac{2\sqrt{106}}{5}$

Solution

Triangle ABC has area $\frac{1}{2}(9)(5) = \frac{45}{2}$. Triangles ABW, WBX, XBY, YBZ, and ZBC have equal bases and altitudes, so the area of each of these small triangles is $\frac{1}{5}\left(\frac{45}{2}\right) = \frac{9}{2}$. Similarly, triangles ADW, WDX, XDY, YDZ, and ZDC each have area $\frac{9}{2}$.

Thus, the shaded region has area $4\left(\frac{9}{2}\right) = 18$.

ANSWER: (C)

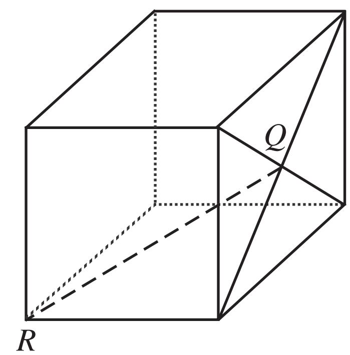
17. If $N = \left(7^{p+4}\right)\left(5^q\right)\left(2^3\right)$ is a perfect cube, where p and q are positive integers, the smallest possible value of $p + q$ is

(**A**) 5 (**B**) 2 (**C**) 8 (**D**) 6 (**E**) 12

Solution

In order for N to be a perfect cube, each prime factor of N must have an exponent that is divisible by 3. Since p and q must be positive integers, the smallest value of p is 2 and the smallest value of q is 3. Thus, the smallest value of $p+q$ is 5. ANSWER: (A)

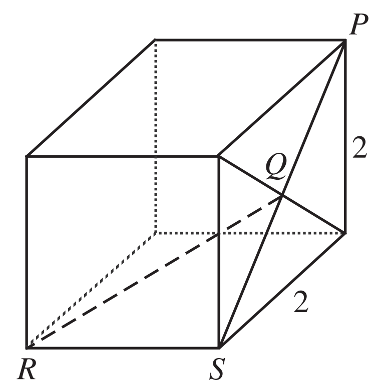
18. Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of QR is



(**A**) 2 (**B**) $\sqrt{8}$ (**C**) $\sqrt{5}$
(**D**) $\sqrt{12}$ (**E**) $\sqrt{6}$

Solution

Label point S as shown. Since each face of the cube is a square with sides of length 2, use the Pythagorean Theorem to find the length of diagonal PS.



$$PS^2 = 2^2 + 2^2$$
$$= 8$$
$$PS = 2\sqrt{2}$$

Then QS has length $\sqrt{2}$, as Q is the midpoint of diagonal PS.
Because we are working with a cube, $\angle QSR = 90^\circ$ and ΔQRS is a right-angled triangle. Use the Pythagorean Theorem in ΔQRS to get

$$QR^2 = 2^2 + \left(\sqrt{2}\right)^2$$
$$= 6$$
$$QR = \sqrt{6}$$

ANSWER: (E)

19. Mr. Anderson has more than 25 students in his class. He has more than 2 but fewer than 10 boys and more than 14 but fewer than 23 girls in his class. How many different class *sizes* would satisfy these conditions?

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 3 (**E**) 4

Solution

Let b represent the number of boys and g represent the number of girls in Mr. Anderson's class. We know that $b+g>25$. We also know $2<b<10$ and $14<g<23$.

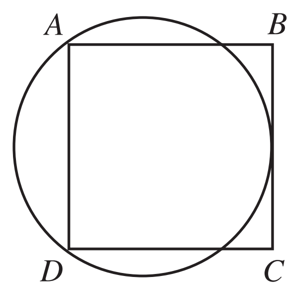
The following pairs (b,g) satisfy all three conditions: (4,22), (5,21), (5,22), (6,20), (6,21), (6,22), (7,19), (7,20), (7,21), (7,22), (8,18), (8,19), (8,20), (8,21), (8,22), (9,17), (9,18), (9,19), (9,20), (9,21), (9,22).

All of the different pairs (b,g) lead to the following different class sizes: 26, 27, 28, 29, 30, 31.

Thus, only 6 different class sizes are possible. ANSWER: (B)

20. Each side of square $ABCD$ is 8. A circle is drawn through A and D so that it is tangent to BC. What is the radius of this circle?

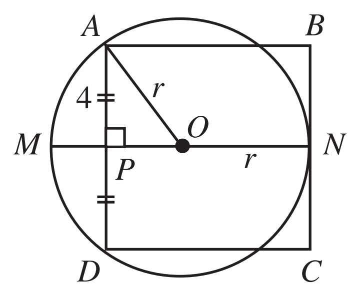
(**A**) 4 (**B**) 5 (**C**) 6
(**D**) $4\sqrt{2}$ (**E**) 5.25



Solution

Let r represent length of the radius and let O represent the centre of the circle. Draw diameter MN that bisects chord AD perpendicularly at P. Join OA.

ΔOAP is a right-angled triangle with $\angle APO = 90°$. The length of AP is 4, since it is half of a side of square $ABCD$. The length of OA is r, and the length of PO is $PN-ON=8-r$.



Using the Pythagorean Theorem we get

$$r^2 = 4^2 + (8-r)^2$$
$$r^2 = 16 + 64 - 16r + r^2$$
$$16r = 80$$
$$r = 5$$

Thus, the radius of the circle is 5. ANSWER: (B)

PART C:

21. When Betty substitutes $x=1$ into the expression ax^3-2x+c its value is -5. When she substitutes $x=4$ the expression has value 52. One value of x that makes the expression equal to zero is

(**A**) 2 (**B**) $\frac{5}{2}$ (**C**) 3 (**D**) $\frac{7}{2}$ (**E**) 4

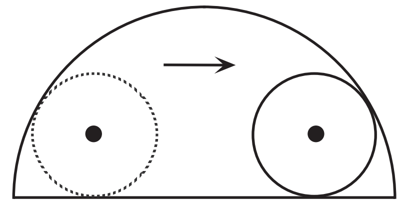
Solution

When $x=1$, we are told that $a(1)^3-2(1)+c=-5$, or $a+c=-3$ (1). Similarly, when $x=4$, $a(4)^3-2(4)+c=52$, or $64a+c=60$ (2). Subtracting equation (1) from equation (2) gives $63a=63$, or $a=1$. Substituting $a=1$ into equation (1) gives $c=-4$. The original expression

$ax^3 - 2x + c$ becomes $x^3 - 2x - 4$. By trial and error, using divisors of 4, when $x = 2$ we get $2^3 - 2(2) - 4 = 0$.

ANSWER: (A)

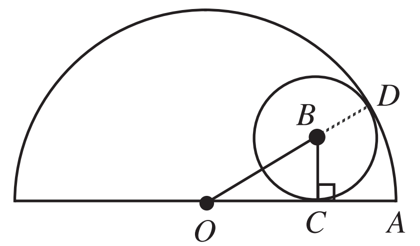
22. A wheel of radius 8 rolls along the diameter of a semicircle of radius 25 until it bumps into this semicircle. What is the length of the portion of the diameter that cannot be touched by the wheel?



(**A**) 8 (**B**) 12 (**C**) 15
(**D**) 17 (**E**) 20

Solution

Draw ΔOBC, where O is the centre of the large circle, B is the centre of the wheel, and C is the point of tangency of the wheel and the diameter of the semicircle. Since BC is a radius of the wheel, $\angle OBC = 90^\circ$ and ΔOBC is right-angled at C.



Extend OB to meet the semicircle at D. Then $BD = BC = 8$, since they are both radii of the wheel, and $OB = 25 - 8 = 17$.

Use the Pythagorean Theorem in ΔOBC to find OC.

$$OC^2 = 17^2 - 8^2$$
$$OC^2 = 225$$
$$OC = 15$$

Then $AC = 25 - 15 = 10$. The length of the portion of the diameter that cannot be touched by the wheel is a length equivalent to $2AC$ or 20.

ANSWER: (E)

23. There are four unequal, positive integers a, b, c, and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$?

(**A**) 13 (**B**) 17 (**C**) 22 (**D**) 33 (**E**) 36

Solution

We are told that $N = 5a + 3b + 5c$ (1) and $N = 4a + 5b + 4c$ (2). Multiply equation (1) by 4 to get $4N = 20a + 12b + 20c$ (3). Similarly, multiply equation (2) by 5 to get $5N = 20a + 25b + 20c$ (4). Subtract equation (3) from equation (4) to get $N = 13b$.

Since N and b are both positive integers with $131 < N < 150$, N must be a multiple of 13. The only possible value for N is 143, when $b = 11$.

Substitute $N = 143$ and $b = 11$ into equation (1) to get

$143 = 5a + 3(11) + 5c$
$110 = 5a + 5c$
$22 = a + c$

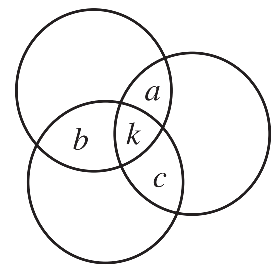
Thus, the value of $a+b+c$ is 22+11=33. ANSWER: (D)

24. Three rugs have a combined area of $200\ \text{m}^2$. By overlapping the rugs to cover a floor area of $140\ \text{m}^2$, the area which is covered by exactly two layers of rug is $24\ \text{m}^2$. What area of floor is covered by three layers of rug?

(**A**) $12\ \text{m}^2$ (**B**) $18\ \text{m}^2$ (**C**) $24\ \text{m}^2$ (**D**) $36\ \text{m}^2$ (**E**) $42\ \text{m}^2$

Solution

Draw the rugs in the following manner, where $a+b+c$ represents the amount of floor covered by exactly two rugs and k represents the amount of floor covered by exactly three rugs. We are told that $a+b+c=24$ (1).



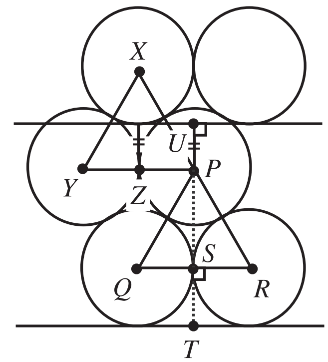
Since the total amount of floor covered when the rugs do not overlap is $200\ \text{m}^2$ and the total covered when they do overlap is $140\ \text{m}^2$, then $60\ \text{m}^2$ of rug is wasted on double or triple layers. Thus, $a+b+c+2k=60$ (2). Subtract equation (1) from equation (2) to get $2k=36$ and solve for $k=18$. Thus, the area of floor covered by exactly three layers of rug is $18\ \text{m}^2$. ANSWER: (B)

25. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

(**A**) 647 (**B**) 1442 (**C**) 1343 (**D**) 1443 (**E**) 1344

Solution

Remove one circle from every second row and shift to form the given configuration. Label the diagram as shown. Since each circle has diameter 1, ΔPQR and ΔPXY are equilateral triangles with sides of length 1.



In ΔPQR, altitude PS bisects side QR. Use the Pythagorean Theorem to find PS.

$$PS^2 = (1)^2 - \left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4}$$
$$PS = \frac{\sqrt{3}}{2}$$

Also, $XZ = \frac{\sqrt{3}}{2}$

Since all radii have length $\frac{1}{2}$, then $PU = \frac{\sqrt{3}}{2} - \frac{1}{2}$ and $TU = \frac{1}{2} + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \sqrt{3}$. This tells us that two rows of circles require a height of $\sqrt{3}$ before a third row begins.

Since $\frac{100}{\sqrt{3}} = 57.7$, we can pack 57 double rows, each containing 100+99=199 circles.

Can we pack one final row of 100 circles? Yes. The square has sides of length 100 and our configuration of 57 double rows requires a height of $57\sqrt{3}$ before the next row begins. Since $100 - 57\sqrt{3} > 1$, and since the circles each have diameter 1, there is room for one final row of 100 circles.

The number of circles used in this new packing is $57(199)+100=11\ 443$

Thus, the maximum number of extra circles that can be packed into the square is

$11\ 443 - 10\ 000 = 1443$. ANSWER: (D)