Trigonometry
Solutions

1. (a) Here $\sin 2\theta = -\frac{1}{2}$. The smallest positive value of 2θ which satisfies this equation is $210°$. Therefore, $\theta = 105°$.

   (b) We know that $\cos^2\theta = 1 - \sin^2\theta$. Substituting this fact into our given equation we obtain

$$\begin{aligned} 2(2\sin^2\theta - 1) &= 8\sin\theta - 5 \\ 4\sin^2\theta - 8\sin\theta + 3 &= 0 \\ (2\sin\theta - 1)(2\sin\theta - 3) &= 0 \\ \sin\theta &= \frac{1}{2}, \frac{3}{2} \end{aligned}$$

   But we know that $|\sin\theta| \le 1$ and so $\sin\theta = \frac{1}{2}$. Therefore, for $-\pi \le \theta \le \pi$, we have $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

2. **Solution 1**

   Let $\theta = \angle AMC$. Therefore, $\angle BMA = 180° - \theta$. Using the cosine law in $\triangle ABM$ gives

$$\begin{aligned} 49 &= 9 + 25 - 30\cos(180° - \theta) \\ 15 &= -30\cos(180° - \theta) \\ \cos(180° - \theta) &= -\frac{1}{2} \end{aligned}$$

   The fact that $\cos(180° - \theta) = -\cos\theta$ gives us $\cos\theta = \frac{1}{2}$.

   Therefore, using the cosine law in $\triangle AMC$ gives

$$AC^2 = 9 + 36 - 36\cos\theta = 45 - 36\left(\frac{1}{2}\right) = 27$$

   Therefore, $AC = 3\sqrt{3}$.

   Solution 2

   Using the cosine law in $\triangle ABM$ gives $9 = 49 + 25 - 70\cos(\angle ABM)$. Thus, $\cos(\angle ABM) = \frac{13}{14}$. Using the cosine law in $\triangle ABC$ gives $AC^2 = 49 + 121 - 154\cos(\angle ABC)$. But $\angle ABC = \angle ABM$ and so $\cos(\angle ABC) = \frac{13}{14}$. Therefore, $AC^2 = 27$ and so $AC = 3\sqrt{3}$.

3. We determine that $\angle BNA = 180° - 47° - 108° = 25°$. Using the sine law in $\triangle BNA$ gives $\frac{BN}{\sin 108°} = \frac{100}{\sin 25°}$ and so $BN = 100\left(\frac{\sin 108°}{\sin 25°}\right)$. But from $\triangle BNM$ we get $\frac{MN}{BN} = \tan 32°$ and so $MN = BN\tan 32°$. Therefore, $MN = 100\left(\frac{\sin 108°}{\sin 25°}\right)\tan 32° \approx 141$ m.

4. Since the area of the rectangle is $\frac{5\pi}{3}$, its height is $\frac{5\pi}{3} \div \frac{\pi}{3} = 5$. Since the cosine graph is symmetrical about the y-axis, $PO = OQ = \frac{\pi}{3} \div 2 = \frac{\pi}{6}$. But $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. Since we are graphing $y = k\cos x$, we see that $k = 5 \div \frac{\sqrt{3}}{2} = \frac{10\sqrt{3}}{3}$.

5. Since the minimum point has a y-coordinate of -2, the amplitude is $a = 2$. Also, since the minimum occurs at $x = \frac{3\pi}{4}$ (rather than $\frac{3\pi}{2}$ where it is found for $\sin x$), $k = \frac{3\pi}{2} \div \frac{3\pi}{4} = 2$. Setting $y = 1$, we obtain that $1 = 2\sin 2x$ and so $\sin 2x = \frac{1}{2}$. Therefore, $2x = \frac{\pi}{6}$ (since we are looking for the intersection that occurs before the first maximum) and so $x = \frac{\pi}{12}$. Thus, $D = \left(\frac{\pi}{12}, 1\right)$.

6. **Solution 1**

   Since one side of each of these triangles is parallel with one of the sides of the square and another side of each of the triangles is parallel to another side of the square, these four triangles are right-angled. We let the length of the side of the triangles opposite θ be a and the length of the side adjacent to θ be b. Then $\tan\theta = \frac{a}{b}$.

   Also, $a - b$ is equal to the length of the sides of the small square which is 3. The area of the large square is equal to the sum of the areas of the triangles along with the area of the small square. Therefore, $4\left(\frac{1}{2}ab\right) + 9 = 89$ and so $b = \frac{40}{a}$. Thus, $a - \frac{40}{a} = 3$ and so $a^2 - 3a - 40 = 0$, which gives $a = 8$ or -5. Now since a is positive, $a = 8$ and $b = 5$. Therefore, $\tan\theta = \frac{8}{5}$.

   Solution 2

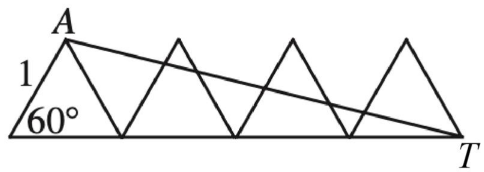
   As in the previous solution, we note that all of the four triangles are right-angled. Each has a hypotenuse of length $\sqrt{89}$, since the area of the large square is 89. If we let the longer of the two legs of the triangle be a, then the length of the other leg is $a - 3$, since the small square has sides of length 3.

   Using the Pythagorean Theorem on this triangle gives $a^2 + (a-3)^2 = 89$ which simplifies to $a^2 - 3a - 40 = 0$. Therefore, $a = 8$ or $a = -5$. But a is positive and so $a = 8$. Therefore, the lengths of the legs of the triangle are 8 and 5 and thus, $\tan\theta = \frac{8}{5}$.

7. Using the Pythagorean theorem, we find that $FA = 2, AC = \sqrt{2}$ and $FC = 2$. The cosine law in $\triangle FAC$ gives

$$\begin{aligned} FC^2 &= FA^2 + AC^2 - 2 \cdot FA \cdot AC \cdot \cos(\angle FAC) \\ 4 &= 4 + 2 - 2 \cdot 2 \cdot \sqrt{2}\cos(\angle FAC) \\ \cos(\angle FAC) &= \frac{1}{2\sqrt{2}}. \end{aligned}$$

8. Consider the diagram below which makes use of four of the small equilateral triangles. Using the cosine law we get $AT^2 = 1^2 + 4^2 - 2(1)(4)\cos 60°$. Therefore, $AT^2 = 13$ and so $AT = \sqrt{13}$.



We can form similar diagrams using four of the small equilateral triangles and again apply the cosine law to get $WA = WT = \sqrt{13}$. Therefore, $\triangle WAT$ is an equilateral triangle with side length $\sqrt{13}$. Using the formula for the area of an equilateral triangle (see the Toolkit for Euclidean Geometry) we have the area of $\triangle WAT = \dfrac{13\sqrt{3}}{4}$.

9. Using the cosine law we get

$$\begin{aligned}
a^2 &= 64 + b^2 - 16b(\cos 60°) \\
&= b^2 - 8b + 64 \\
&= (b-4)^2 + 48 \\
a^2 - (b-4)^2 &= 48 \\
(a+b-4)(a-b+4) &= 48
\end{aligned}$$

Since 48 is positive, the factors must be both positive or both negative. The sum of the two factors is $2a$ and since a is a positive integer, the sum of the factors must be even and positive. Therefore, they must be both even or both odd and they must both be positive. Since 48 is even, both of the factors must be even. The even-even factorizations of 48 are $2 \cdot 24, 4 \cdot 12, 6 \cdot 8, 8 \cdot 6, 12 \cdot 4, 24 \cdot 2$.

We have already discovered that a is the sum of the factors divided by 2. The difference of the factors is $a + b - 4 - (a - b + 4) = 2b - 8$ and so $b = \dfrac{d}{2} + 4$, where d is the difference of factors.

factors	sum	difference, d	a	b
$2, 24$	26	-22	13	-7
$4, 12$	16	-8	8	0
$6, 8$	14	-2	7	3
$8, 6$	14	2	7	5
$12, 4$	16	8	8	8
$24, 2$	26	22	13	15

Since b must be positive, the possible values for a and b are $(a, b) = (7, 3), (7, 5), (8, 8), (13, 15)$.