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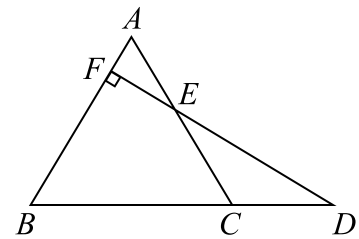
December 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *1997 Cayley Contest, Question 22*

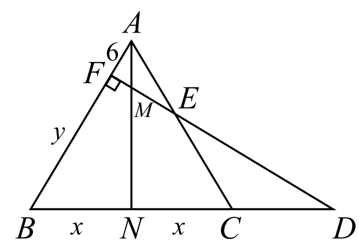
In the diagram, $\triangle ABC$ is equilateral, $BC = 2CD$, $AF = 6$, and DEF is perpendicular to AB. What is the area of quadrilateral $FBCE$?



(A) $144\sqrt{3}$ **(B)** $138\sqrt{3}$ **(C)** $126\sqrt{3}$
(D) $108\sqrt{3}$ **(E)** $66\sqrt{3}$

Solution

Drop a perpendicular from A to BC, and label as shown.



Since $\triangle ABC$ is equilateral, $BN = NC = CD$. Let $BN = x$ and $BF = y$. Then $6 + y = 2x$ or $y = 2x - 6$.
Also, $\angle FAM = 30°$, and $\triangle AMF$ is a 30°-60°-90° triangle with sides in the ratio $1 : \sqrt{3} : 2$. Thus, $AM = \frac{2}{\sqrt{3}}AF = 4\sqrt{3}$ and $FM = \frac{1}{2}AM = 2\sqrt{3}$.
$\triangle DBF$ is also a 30°-60°-90° triangle because it is right-angled at F and has a 60° angle at B.
Therefore, $3x : y = 2 : 1$ or $3x = 2y$.
Since $y = 2x - 6$, then $3x = 2(2x - 6)$ or $3x = 4x - 12$ and so $x = 12$ which gives $y = 18$.
$\triangle ABN$ is also a 30°-60°-90° triangle which gives $AN = \sqrt{3}BN = \sqrt{3}x = 12\sqrt{3}$.
The area of $\triangle ABC$ is thus $\frac{1}{2}(BC)(AN) = \frac{1}{2}(24)(12\sqrt{3}) = 144\sqrt{3}$.
$\triangle AEF$ is also a 30°-60°-90° triangle which gives $FE = \sqrt{3}AF = 6\sqrt{3}$.
The area of $\triangle AEF$ is thus $\frac{1}{2}(AF)(FE) = \frac{1}{2}(6)(6\sqrt{3}) = 18\sqrt{3}$.
Finally, the area of quadrilateral $FECB$ equals the area of $\triangle ABC$ minus the area of $\triangle AFE$, or $144\sqrt{3} - 18\sqrt{3} = 126\sqrt{3}$.

ANSWER: (C)

2. *2012 Canadian Senior Mathematics Contest, Question B2*
Consider the equation $x^2 - 2y^2 = 1$, which we label ①. There are many pairs of positive integers (x, y) that satisfy equation ①.

(a) Determine a pair of positive integers (x, y) with $x \leq 5$ that satisfies equation ①.

(b) Determine a pair of positive integers (u, v) such that

$$(3 + 2\sqrt{2})^2 = u + v\sqrt{2}$$

and show that (u, v) satisfies equation ①.

(c) Suppose that (a, b) is a pair of positive integers that satisfies equation ①. Suppose also that (c, d) is a pair of positive integers such that

$$(a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}$$

Show that (c, d) satisfies equation ①.

(d) Determine a pair of positive integers (x, y) with $y > 100$ that satisfies equation ①.

Solution

(a) With some trial and error, we find that the pair $(x, y) = (3, 2)$ satisfies the given equation, since $3^2 - 2(2^2) = 1$.

(b) Expanding $(3 + 2\sqrt{2})^2$, we obtain

$$u + v\sqrt{2} = (3 + 2\sqrt{2})(3 + 2\sqrt{2}) = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 = 17 + 12\sqrt{2}$$

Therefore, $(u, v) = (17, 12)$ satisfies the equation $(3 + 2\sqrt{2})^2 = u + v\sqrt{2}$.
Furthermore, if $(u, v) = (17, 12)$, then $u^2 - 2v^2 = 17^2 - 2(12^2) = 289 - 2(144) = 1$.
Thus, $(u, v) = (17, 12)$ satisfies equation ①, as required.

(c) Since (a, b) satisfies ①, then $a^2 - 2b^2 = 1$.
Since $(a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}$, then

$$c + d\sqrt{2} = 3a + 2a\sqrt{2} + 3b\sqrt{2} + 2b(2) = (3a + 4b) + (2a + 3b)\sqrt{2}$$

and so $(c, d) = (3a + 4b, 2a + 3b)$.

(It is not hard to see that *if* $(c, d) = (3a + 4b, 2a + 3b)$, *then* we have

$$c + d\sqrt{2} = (3a + 4b) + (2a + 3b)\sqrt{2}$$

To formally justify that $c + d\sqrt{2} = (3a + 4b) + (2a + 3b)\sqrt{2}$ implies $(c, d) = (3a + 4b, 2a + 3b)$, we can rearrange the equation to obtain

$$c - 3a - 4b = (2a + 3b - d)\sqrt{2}$$

The left side of this equation is an integer. If $2a + 3b - d \neq 0$, the right side of this equation is irrational (because $\sqrt{2}$ is irrational), so cannot equal an integer. Thus, $2a + 3b - d = 0$ or $d = 2a + 3b$. This implies that $c - 3a - 4b = 0$ or $c = 3a + 4b$.)

To show that (c, d) satisfies equation ①, we need to show that $c^2 - 2d^2 = 1$:

$$\begin{aligned}
c^2 - 2d^2 &= (3a + 4b)^2 - 2(2a + 3b)^2 \\
&= (9a^2 + 24ab + 16b^2) - 2(4a^2 + 12ab + 9b^2) \\
&= 9a^2 + 24ab + 16b^2 - 8a^2 - 24ab - 18b^2 \\
&= a^2 - 2b^2 \\
&= 1 \qquad (\text{since } a^2 - 2b^2 = 1)
\end{aligned}$$

Therefore, (c, d) satisfies equation ①, as required.

(d) From (c), we know that if (a, b) is a solution to equation ①, then $(c, d) = (3a + 4b, 2a + 3b)$ is also a solution to equation ①.
From (b), we know that $(17, 12)$ is a solution to equation ①.
We use these two facts together.
Since $(17, 12)$ is a solution, then $(3(17) + 4(12), 2(17) + 3(12)) = (99, 70)$ is a solution.
Since $(99, 70)$ is a solution, then $(3(99) + 4(70), 2(99) + 3(70)) = (577, 408)$ is a solution, which has $y > 100$.
We can verify that $577^2 - 2(408^2) = 1$, as required.

3. *2003 Fryer Contest, Question 2*
In a game, Xavier and Yolanda take turns calling out whole numbers. The first number called must be a whole number between and including 1 and 9. Each number called after the first must be a whole number which is 1 to 10 greater than the previous number called.

(a) The first time the game is played, the person who calls the number 15 is the winner. Explain why Xavier has a winning strategy if he goes first and calls 4.

(b) The second time the game is played, the person who calls the number 50 is the winner. If Xavier goes first, how does he guarantee that he will win?

Solution

(a) If Xavier goes first and calls 4, then on her turn Yolanda can call any number from 5 to 14, since her number has to be from 1 to 10 greater than Xavier's.
But if Yolanda calls a number from 5 to 14, then Xavier can call 15 on his next turn, since 15 is from 1 to 10 bigger than any of the possible numbers that Yolanda can call. So Xavier can call 15 on his second turn no matter what Yolanda calls, and is thus always guaranteed to win.

(b) In (a), we saw that if Xavier calls 4, then he can guarantee that he can call 15.
Using the same argument, shifting all of the numbers up, to guarantee that he can call 50, he should call 39 on his previous turn.
(In this case, Yolanda can call any whole number from 40 to 49, and in any of these cases Xavier can call 50, since 50 is no more than 10 greater than any of these numbers.) In a similar way, to guarantee that he can call 39, he should call 28 on his previous turn, which he can do for the same reasons as above.
To guarantee that he can call 28, he should call 17 on his previous turn. To guarantee that he can call 17, he should call 6 on his previous turn, which could be his first turn.
Therefore, Xavier's winning strategy is to call 6 on his first turn, 17 on his second turn, 28 on his third turn, 39 on his fourth turn, and 50 on his last turn.
At each step, we are using the fact that Xavier can guarantee that his number on one turn is 11 greater than his number on his previous turn. This is because Yolanda adds 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 to his previous number, and he can then correspondingly add 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1 to her number, for a total of 11 in each case.

4. *2013 Euclid Contest, Question 7b*

Determine all linear functions $f(x) = ax + b$ such that if $g(x) = f^{-1}(x)$ for all values of x, then $f(x) - g(x) = 44$ for all values of x. (Note: f^{-1} is the inverse function of f.)

Solution

Since $f(x) = ax + b$, we can determine an expression for $g(x) = f^{-1}(x)$ by letting $y = f(x)$ and to obtain $y = ax + b$. We then interchange x and y to obtain $x = ay + b$ which we solve for y to obtain $ay = x - b$ or $y = \dfrac{x}{a} - \dfrac{b}{a}$.

Therefore, $f^{-1}(x) = \dfrac{x}{a} - \dfrac{b}{a}$.

Note that $a \neq 0$. (This makes sense since the function $f(x) = b$ has a graph which is a horizontal line, and so cannot be invertible.)

Therefore, the equation $f(x) - g(x) = 44$ becomes $(ax + b) - \left(\dfrac{x}{a} - \dfrac{b}{a}\right) = 44$ or

$$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 44 = 0x + 44$$

Since this equation is true for all x, then the coefficients of the linear expression on the left side must match the coefficients of the linear expression on the right side.

Therefore, $a - \dfrac{1}{a} = 0$ and $b + \dfrac{b}{a} = 44$.

From the first of these equations, we obtain $a = \dfrac{1}{a}$ or $a^2 = 1$, which gives $a = 1$ or $a = -1$.

If $a = 1$, the equation $b + \dfrac{b}{a} = 44$ becomes $b + b = 44$, which gives $b = 22$.

If $a = -1$, the equation $b + \dfrac{b}{a} = 44$ becomes $b - b = 44$, which is not possible.

Therefore, we must have $a = 1$ and $b = 22$, and so $f(x) = x + 22$.

5. *2016 Canadian Intermediate Mathematics Contest, Question A4*

Dina has a calculating machine, labelled f, that takes one number as input and calculates an output. The machine f calculates its output by multiplying its input by 2 and then subtracting 3. For example, if Dina inputs 2.16 into f, the output is 1.32. If Dina inputs a number x into f, she gets a first output which she then inputs back into f to obtain a second output, which is -35. What is the value of x?

Solution

Dina's machine works by taking an input, multiplying by 2 and then subtracting 3.

If we have the output and want to obtain the input, we must reverse these operations and so we take the output, add 3 and then divide by 2.

Starting with the second output -35, we obtain $-35 + 3 = -32$ and then $\frac{-32}{2} = -16$, which was the second input.

Since the second input was the first output, then the first output was -16.

Starting with the first output -16, we obtain $-16 + 3 = -13$ and then $\frac{-13}{2}$, which was the first input.

Therefore, the first input was $-\frac{13}{2}$ or -6.5.