The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

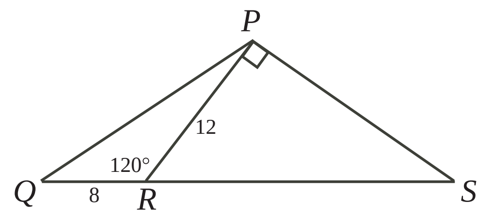
From the archives of the CEMC

August 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *2008 Fermat Contest, Question 19*

In the diagram, R is on QS and $QR = 8$. Also, $PR = 12$, $\angle PRQ = 120^\circ$, and $\angle RPS = 90^\circ$. What is the area of $\triangle QPS$?

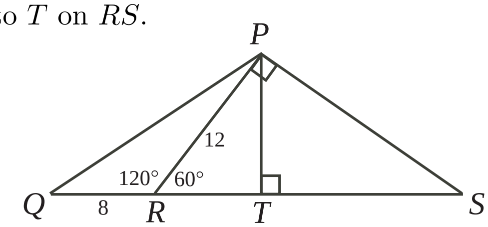


(A) $72\sqrt{3}$ **(B)** 72 **(C)** 36

(D) $60\sqrt{3}$ **(E)** $96\sqrt{3}$

Solution

Since $\angle QRP = 120^\circ$ and QRS is a straight line, then $\angle PRS = 180^\circ - 120^\circ = 60^\circ$.
Since $\angle RPS = 90^\circ$, then $\triangle SRP$ is a 30°-60°-90° triangle.
Therefore, $RS = 2PR = 2(12) = 24$.
Drop a perpendicular from P to T on RS.



Since $\angle PRT = 60^\circ$ and $\angle PTR = 90^\circ$, then $\triangle PRT$ is also a 30°-60°-90° triangle.
Therefore, $PT = \frac{\sqrt{3}}{2}PR = 6\sqrt{3}$.
Consider $\triangle QPS$. We may consider QS as its base with height PT.
Thus, its area is $\frac{1}{2}(6\sqrt{3})(8 + 24) = 96\sqrt{3}$.

ANSWER: (E)

2. *1988 Gauss Contest, Grade 7, Question 24*

A set of positive integers, each of which is different, has a sum of 329 and an average of 47. If one of the integers is 97, then the largest possible integer that could appear in the set is

(A) 97 **(B)** 211 **(C)** 217 **(D)** 227 **(E)** 228

Solution

A set with a sum of 329 and an average of 47 contains $329 \div 47 = 7$ integers.
When the integer 97 is removed, there are 6 integers with a sum of 232.
To make one integer as large as possible, the other 5 positive integers must be as small as possible, thus 1, 2, 3, 4, 5.
Therefore, the largest possible integer that could appear in the set is $232 - 1 - 2 - 3 - 4 - 5 = 217$.

ANSWER: (C)

3. *2000 Gauss Contest, Grade 7, Question 6*

In the addition shown, a digit, either the same or different, can be placed in each of the two boxes. What is the sum of the two missing digits?

$$\begin{array}{ccccc} & & 8 & 6 & 3 \\ & & \square & 9 & 1 \\ + & & 7 & \square & 8 \\ \hline & 2 & 1 & 8 & 2 \end{array}$$

(A) 9 **(B)** 11 **(C)** 13
(D) 3 **(E)** 7

Solution

Adding in the units (ones) column gives us $3 + 1 + 8 = 12$.
This means a carry of 1 into the tens column since $12 = 1 \times 10 + 2$.
In the tens column, the sum of the carry plus the digits must have a units digit of 8.
Thus, $1 + 6 + 9 + \square$ has a units digit of 8, and so 2 is the missing digit.
This gives a sum of 18 in the tens column, and so means a carry of 1 into the hundreds column.
In the tens column, the sum of the carry plus the digits must equal 21.
Thus, $1 + 8 + \square + 7 = 21$, and so 5 is the missing digit.
The two missing digits are 2 and 5 giving a sum of 7.

ANSWER: (E)

4. *2005 Cayley Contest, Question 25*

The positive integers a, b, and c satisfy $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$. The sum of all possible values of $a \leq 100$ is

(A) 315 **(B)** 615 **(C)** 680 **(D)** 555 **(E)** 620

Solution

The first challenge in this problem is to find one set of numbers a, b, c that actually works.
Since this looks a bit similar to the Pythagorean Theorem, we can start with $3^2 + 4^2 = 5^2$ and try to manipulate this.
If we divide both sides by the least common multiple of 3^2, 4^2 and 5^2, which is $(3 \times 4 \times 5)^2 = 60^2$, we then obtain $\frac{3^2}{60^2} + \frac{4^2}{60^2} = \frac{5^2}{60^2}$ or $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$.
This gives us two possible triples: $(a, b, c) = (20, 15, 12)$ and $(a, b, c) = (15, 20, 12)$ (so two possible values for a so far).
How can we get more? We can multiply the equation $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$ by reciprocals of perfect squares.
Multiplying by $\frac{1}{2^2}$, we get $\frac{1}{40^2} + \frac{1}{30^2} = \frac{1}{24^2}$.
Multiplying by $\frac{1}{3^2}$, we get $\frac{1}{60^2} + \frac{1}{45^2} = \frac{1}{36^2}$.
Multiplying by $\frac{1}{4^2}$, we get $\frac{1}{80^2} + \frac{1}{60^2} = \frac{1}{48^2}$.
Multiplying by $\frac{1}{5^2}$, we get $\frac{1}{100^2} + \frac{1}{75^2} = \frac{1}{60^2}$.
Multiplying by $\frac{1}{6^2}$, we get $\frac{1}{120^2} + \frac{1}{90^2} = \frac{1}{72^2}$.
Multiplying by $\frac{1}{7^2}$, we get $\frac{1}{140^2} + \frac{1}{105^2} = \frac{1}{84^2}$.
At this point, the strategy will no longer work, since we are only looking for values of $a \leq 100$.
So far, the possible values of a are (from looking at each denominator of the left side of each the equations here): 20, 15, 40, 30, 60, 45, 80, 100, 75, 90. (Notice that 60 doesn't appear twice in the list!) The sum of these numbers is 555.

Can we find more starting equations by beginning with a different Pythagorean triple?
If we start with $5^2 + 12^2 = 13^2$ and divide both sides by the least common multiple of 5^2, 12^2 and 13^2 (ie. $(5 \times 12 \times 13)^2 = 780^2$), we get $\frac{1}{156^2} + \frac{1}{65^2} = \frac{1}{60^2}$ which gives us 65 as another possible value of a.
Therefore, our running total for values of a is $555 + 65 = 620$.

We can't generate more possible values for a using $\frac{1}{156^2} + \frac{1}{65^2} = \frac{1}{60^2}$ since multiplying both sides by the reciprocal of any perfect square will make both of a and b at least 130, so bigger than 100.

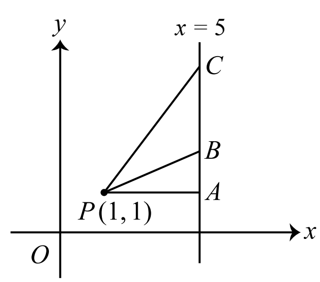
Can we use $6^2 + 8^2 = 10^2$? Here, the least common multiple of 6^2, 8^2 and 10^2 is 120^2, and dividing by 120^2 gives us $\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}$, which we have already used.
Can we use any other Pythagorean triple? No, since any other Pythagorean triple is at least as big as 7-24-25, and so the smallest possible denominator that we will get on the left side by using this technique is $(7 \times 25)^2 = 175^2$, which would give an a larger than 100.
Also, any triple (a, b, c) that actually works does come from a Pythagorean triple, since we can multiply both sides of $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ by $(abc)^2$ to get $(bc)^2 + (ac)^2 = (ab)^2$.
So every possible triple (a, b, c) comes from a Pythagorean triple, and no Pythagorean triples give any more allowable values of a, so we have found them all.
Therefore, the sum of all possible values of $a \leq 100$ is 620.

ANSWER: (E)

5. *2017 Cayley Contest, Question 19*
Two lines with slopes $\frac{1}{4}$ and $\frac{5}{4}$ intersect at (1,1). What is the area of the triangle formed by these two lines and the vertical line $x = 5$?

(A) 5 **(B)** 10 **(C)** 8 **(D)** 12 **(E)** 15

Solution
We start by drawing a graph that includes the point $(1, 1)$, the lines with slopes $\frac{1}{4}$ and $\frac{5}{4}$ that pass through this point, the vertical line with equation $x = 5$, and the horizontal line $y = 1$ (which passes through $(1, 1)$ and is perpendicular to the vertical line with equation $x = 5$).
We label the various points of intersection (A, B, C) as shown.



We want to determine the area of $\triangle PBC$.
Since P has coordinates $(1, 1)$ and A has coordinates $(5, 1)$, then $PA = 4$.
Since the slope of PB is $\frac{1}{4}$ and $PA = 4$, then thinking about slope as "rise over run", we see that $AB = 1$.
Since the slope of PC is $\frac{5}{4}$ and $PA = 4$, then $AC = 5$.
Since $AC = 5$ and $AB = 1$, then $BC = AC - AB = 5 - 1 = 4$.
We can view $\triangle PBC$ as having base BC and perpendicular height PA. (This is because the length of PA is the perpendicular distance from the line through B and C to the point P.)
Therefore, the area of this triangle is $\frac{1}{2}(4)(4)$ which equals 8.
Thus, the area of $\triangle PBC$ is $10 - 2 = 8$.

ANSWER: (C)