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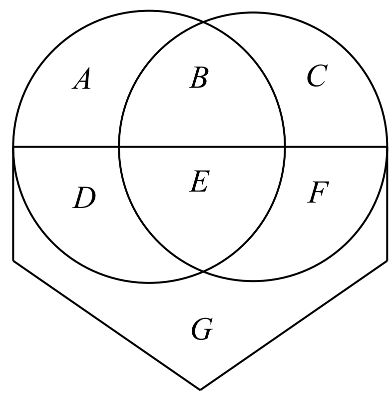
July 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *1989 Pascal Contest, Question 15*

   Two circles, each with a radius of one, and a pentagon overlap as shown in the diagram. The bounded regions labelled A, B, C, D, E, F, and G are of equal area. The area of the pentagon is

   (A) 1 **(B)** $\frac{4}{7}$ **(C)** $\frac{\pi}{4}$
   (D) π **(E)** 2π



Solution

The area of each of the circles is $\pi(1)^2 = \pi$.
Each of the circles is made up of four regions of equal area.
Thus, each of the regions A, B, C, D, E, F, and G has area $\frac{1}{4}\pi$.
Therefore, the area of the pentagon is $D + E + F + G = 4(\frac{1}{4}\pi) = \pi$.

ANSWER: (D)

2. *2014 Fermat Contest, Question 22*

Jillian drives along a straight road that goes directly from her house (J) to her Grandfather's house (G). Some of this road is on flat ground and some is downhill or uphill. Her car travels downhill at 99 km/h, on flat ground at 77 km/h, and uphill at 63 km/h. It takes Jillian 3 hours and 40 minutes to drive from J to G. It takes her 4 hours and 20 minutes to drive from G to J. The distance between J and G, in km, is

(A) $318\frac{2}{3}$ **(B)** 324 **(C)** 308 **(D)** $292\frac{3}{5}$ **(E)** $292\frac{1}{9}$

Solution

As Jillian drives from J to G, suppose that she drives x km uphill, y km on flat ground, and z km downhill.

This means that when she drives from G to J, she will drive z km uphill, y km on flat ground, and x km downhill. This is because downhill portions become uphill portions on the return trip, while uphill portions become downhill portions on the return trip.

We are told that Jillian drives at 77 km/h on flat ground, 63 km/h uphill, and 99 km/h downhill.

Since time equals distance divided by speed, then on her trip from J to G, her time driving uphill is $\dfrac{x}{63}$ hours, her time driving on flat ground is $\dfrac{y}{77}$ hours, and her time driving downhill is $\dfrac{z}{99}$ hours.

Since it takes her 3 hours and 40 minutes (which is $3\dfrac{2}{3}$ or $\dfrac{11}{3}$ hours), then

$$\frac{x}{63}+\frac{y}{77}+\frac{z}{99}=\frac{11}{3}$$

A similar analysis of the return trip gives

$$\frac{x}{99}+\frac{y}{77}+\frac{z}{63}=\frac{13}{3}$$

We are asked for the total distance from J to G, which equals $x+y+z$ km. Therefore, we need to determine $x+y+z$.

We add the two equations above and simplify to obtain

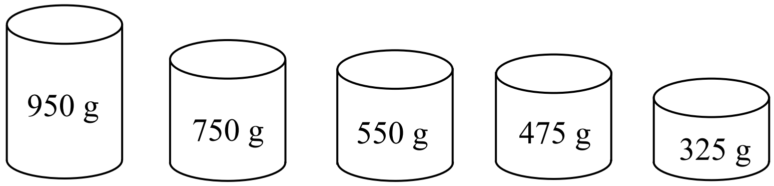
$$\begin{aligned}
\frac{x}{63}+\frac{x}{99}+\frac{y}{77}+\frac{y}{77}+\frac{z}{99}+\frac{z}{63} &= \frac{24}{3}\\
x\left(\frac{1}{63}+\frac{1}{99}\right)+y\left(\frac{1}{77}+\frac{1}{77}\right)+z\left(\frac{1}{99}+\frac{1}{63}\right) &= 8\\
x\left(\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{1}{9\cdot 11}+\frac{1}{7\cdot 9}\right) &= 8\\
x\left(\frac{11}{7\cdot 9\cdot 11}+\frac{7}{7\cdot 9\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{7}{7\cdot 9\cdot 11}+\frac{11}{7\cdot 9\cdot 11}\right) &= 8\\
x\left(\frac{18}{7\cdot 9\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{18}{7\cdot 9\cdot 11}\right) &= 8\\
x\left(\frac{2}{7\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{2}{7\cdot 11}\right) &= 8\\
\frac{2}{77}(x+y+z) &= 8
\end{aligned}$$

Thus, $x+y+z=\dfrac{77}{2}\cdot 8=77\cdot 4=308$.

Finally, the distance from J to G is 308 km.

ANSWER: (C)

3. The five tins shown contain either coffee or cocoa or powdered milk.



There is twice as much coffee as cocoa by total weight. No three tins contain the same item. The only tin containing cocoa is the

(A) 950 g tin **(B)** 750 g tin **(C)** 550 g tin **(D)** 475 g tin **(E)** 325 g tin

Solution

Since no three tins contain the same item and only one tin contains cocoa, then there are two tins of coffee since there cannot be three tins of coffee or one tin of coffee (the latter of which would give three tins of milk).

The tin containing cocoa has a weight that is half the weight of two tins of coffee combined.

We note that $950 + 550 = 1500 = 2(750)$, and so the tin containing cocoa could have weight 750 g.

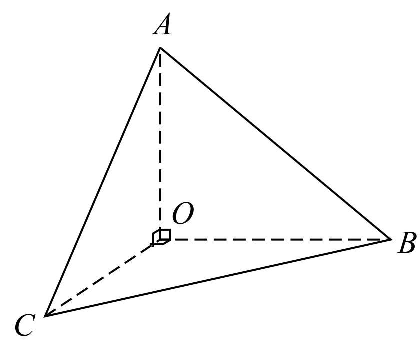
Further, any of the remaining tins, as there are not two tins whose combined weight is 1900 g, 1100 g, 950 g, or 650 g.

Thus, the tin containing cocoa has weight 750 g.

ANSWER: (B)

4. *1993 Grade 11 Invitational Mathematics Challenge, Question 5*

In the tetrahedron shown in the diagram, angles COA, AOB and COB are right angles. The three triangles meeting at O have areas of 6, $\sqrt{39}$, and 5 units. Determine the area of $\triangle ABC$.



Solution

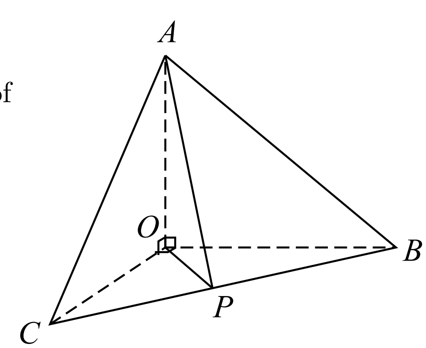
There is a three-dimensional analogue of the Pythagorean Theorem that says that, if $OABC$ is a tetrahedron with $\angle AOB = \angle COA = \angle BOC = 90°$, then the square of the area of $\triangle ABC$ equals the sum of the squares of the areas of $\triangle AOB$, $\triangle COA$, and $\triangle BOC$.

Assuming that this is true, we obtain that the square of the area of $\triangle ABC$ equals $6^2 + (\sqrt{39})^2 + 5^2$ which equals $36 + 39 + 25$ which is 100, and so the area of $\triangle ABC$ is 10.

Why is this result true?

Draw an altitude in $\triangle COB$ from O to P on CB.

Since AO and OP are both perpendicular to CB, then the plane of $\triangle AOP$ is perpendicular to CB, so AP is perpendicular to CB.



Therefore,

$$\begin{aligned}|\triangle ABC|^2 &= \left(\tfrac{1}{2}\cdot CB\cdot AP\right)^2\\ &= \tfrac{1}{4}\cdot CB^2\cdot AP^2\\ &= \tfrac{1}{4}(CO^2+BO^2)(AO^2+OP^2)\\ &\quad\text{(using the Pythagorean Theorem in } \triangle COB \text{ and } \triangle AOP)\\ &= \tfrac{1}{4}\cdot CO^2\cdot AO^2+\tfrac{1}{4}\cdot BO^2\cdot AO^2+\tfrac{1}{4}\cdot(CO^2+BO^2)\cdot OP^2\\ &= \left(\tfrac{1}{2}\cdot AO\cdot CO\right)^2+\left(\tfrac{1}{2}\cdot AO\cdot BO\right)^2+\left(\tfrac{1}{2}\cdot CB\cdot OP\right)^2\\ &= |\triangle AOC|^2+|\triangle AOB|^2+|\triangle BOC|^2\end{aligned}$$

as required.

5. *2017 Fryer Contest, Question 2*

By finding a common denominator, we see that $\frac{1}{3}$ is greater than $\frac{1}{7}$ because $\frac{7}{21} > \frac{3}{21}$.

Similarly, we see that $\frac{1}{3}$ is less than $\frac{1}{2}$ because $\frac{2}{6} < \frac{3}{6}$.

(a) Determine the integer n so that $\frac{n}{40}$ is greater than $\frac{1}{5}$ and less than $\frac{1}{4}$.

(b) Determine all possible integers m so that $\frac{m}{8}$ is greater than $\frac{1}{3}$ and $\frac{m+1}{8}$ is less than $\frac{2}{3}$.

(c) Fiona calculates her *win ratio* by dividing the number of games that she has won by the total number of games that she has played. At the start of a weekend, Fiona has played 30 games, has w wins, and her win ratio is greater than 0.5. During the weekend, she plays five games and wins three of these games. At the end of the weekend, Fiona's win ratio is less than 0.7. Determine all possible values of w.

Solution

(a) Expressing $\frac{1}{5}$ and $\frac{1}{4}$ with a common denominator of 40, we get $\frac{1}{5} = \frac{8}{40}$ and $\frac{1}{4} = \frac{10}{40}$.
We require that $\frac{n}{40} > \frac{8}{40}$ and $\frac{n}{40} < \frac{10}{40}$, thus $n > 8$ and $n < 10$.
The only integer n that satisfies both of these inequalities is $n = 9$.

(b) Expressing $\frac{m}{8}$ and $\frac{1}{3}$ with a common denominator of 24, we require $\frac{3m}{24} > \frac{8}{24}$ and so $3m > 8$ or $m > \frac{8}{3}$. Since $\frac{8}{3} = 2\frac{2}{3}$ and m is an integer, then $m \geq 3$.
Expressing $\frac{m+1}{8}$ and $\frac{2}{3}$ with a common denominator of 24, we require $\frac{3(m+1)}{24} < \frac{16}{24}$ or $3m+3 < 16$ or $3m < 13$, and so $m < \frac{13}{3}$.
Since $\frac{13}{3} = 4\frac{1}{3}$ and m is an integer, then $m \leq 4$.
The integer values of m which satisfy $m \geq 3$ and $m \leq 4$ are $m = 3$ and $m = 4$.

(c) At the start of the weekend, Fiona has played 30 games and has w wins, so her win ratio is $\frac{w}{30}$.
Fiona's win ratio at the start of the weekend is greater than $0.5 = \frac{1}{2}$, and so $\frac{w}{30} > \frac{1}{2}$.
Since $\frac{1}{2} = \frac{15}{30}$, then we get $\frac{w}{30} > \frac{15}{30}$, and so $w > 15$.

During the weekend Fiona plays five games giving her a total of $30 + 5 = 35$ games played.
Since she wins three of these games, she now has $w + 3$ wins, and so her win ratio is $\dfrac{w+3}{35}$.
Fiona's win ratio at the end of the weekend is less than $0.7 = \dfrac{7}{10}$, and so $\dfrac{w+3}{35} < \dfrac{7}{10}$.
Rewriting this inequality with a common denominator of 70, we get $\dfrac{2(w+3)}{70} < \dfrac{49}{70}$ or $2(w+3) < 49$ or $2w + 6 < 49$ or $2w < 43$, and so $w < \dfrac{43}{2}$.
Since $\dfrac{43}{2} = 21\dfrac{1}{2}$ and w is an integer, then $w \leq 21$.
The integer values of w which satisfy $w > 15$ and $w \leq 21$ are $w = 16, 17, 18, 19, 20, 21$.