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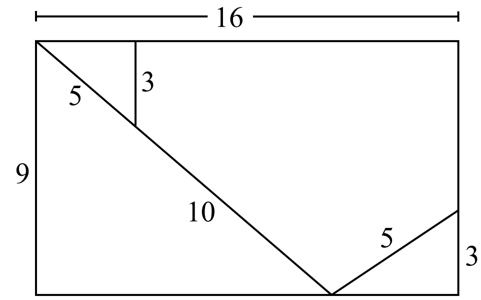
January 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

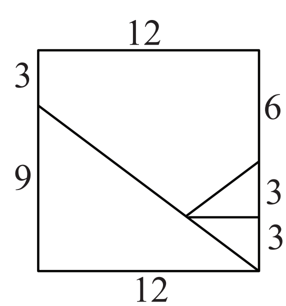
1. *1963 Junior Mathematics Contest, Question 13*
   When the 16 by 9 rectangle in the diagram is cut in the manner shown, the pieces can form a square of perimeter:

   (A) 50 **(B)** 48 **(C)** 32
   (D) 40 **(E)** 36



Solution 1
The rectangle has area $9 \times 16 = 144$.
The square created with the pieces of the rectangle will have the same area.
Thus, the side length of the square will be $\sqrt{144} = 12$ and its perimeter will be $4 \times 12 = 48$.

Solution 2
Using the Pythagorean Theorem on the two small triangles with sides 3 and 5, we find that the missing leg in each case is of length $\sqrt{5^2 - 3^2} = 4$.
Using this information, we find that the side length of the pentagon which constitutes part of the top of the rectangle is $16 - 4 = 12$.
Applying the Pythagorean Theorem to the large triangle with sides 9 and 15, we find that the missing leg is of length $\sqrt{15^2 - 9^2} = 12$.
Using this information and rearranging the pieces in the diagram given, we can create the square below:



Thus we have a square of perimeter $4 \times 12 = 48$.

ANSWER: (B)

2. *1974 Gauss Contest, Question 17*
Nine coins have a value of \$1.35. If the coins are quarters and dimes, the number of dimes is
(A) 2 **(B)** 5 **(C)** 7 **(D)** 1 **(E)** 6

Solution
The dimes and quarters number x and $9-x$ for some non-negative integer x.
The values of the dimes and the quarters are $10x$ cents and $25(9-x)$ cents, respectively.
We have a total of \$1.35, or 135 cents. Therefore,

$$\begin{aligned} 10x + 25(9-x) &= 135 \\ 225 - 135 &= 15x \\ 90 &= 15x \\ 6 &= x \end{aligned}$$

There are 6 dimes.

ANSWER: (E)

3. *1980 Euclid Contest, Question A10*
If n is the number of digits in 2^{3217}, then
(A) $900 \leq n \leq 950$ **(B)** $965 \leq n \leq 990$ **(C)** $1000 \leq n \leq 1050$
(D) $1070 \leq n \leq 1075$ **(E)** $n > 1075$

Solution
We note that $2^{10} = 1024 > 10^3$ and $2^7 = 128 > 10^2$.
Therefore, $2^{3217} = (2^{10})^{321} \times 2^7 > (10^3)^{321} \times 2^7 > 10^{963} \times 10^2 = 10^{965}$
Thus, $2^{3217} > 10^{965}$
Thus 2^{3217} will contain at least 965 digits. (In fact, 2^{3217} has at least 966 digits because 10^{965} has 966 digits.)
Now, note that $2^{13} = 8192 < 10^4$ and $2^6 = 64 < 10^2$.
Therefore, $2^{3217} = (2^{13})^{247} \times 2^6 < (10^4)^{247} \times 2^6 < 10^{988} \times 10^2 = 10^{990}$
Thus, $2^{3217} < 10^{990}$
Thus, 2^{3217} will contain at most 990 digits.
Thus, $965 \leq n \leq 990$.

ANSWER: (B)

4. *1970 Ontario Senior Mathematics Problems Competition, Question 8*
Prove that the equation $6x^2 + 2y^2 = z^2$ has no solution in integers x, y, z, except for $x = y = z = 0$.

Solution
$(x, y, z) = (0, 0, 0)$ solves the original equation. Suppose next that at least one of x, y, z is non-zero.
If x, y, and z have a greatest common divisor p, then $x = pX, y = pY, z = pZ$ for some integers X, Y, Z with no common divisor greater than 1.
If (x, y, z) is a solution, then

$$\begin{aligned} 6(pX)^2 + 2(pY)^2 &= (pZ)^2 \\ 6p^2X^2 + 2p^2Y^2 &= p^2Z^2 \\ 6X^2 + 2Y^2 &= Z^2 \end{aligned}$$

Since each solution with a common divisor larger than 1 corresponds to a solution with greatest common divisor of 1, we can look only for solutions with greatest common divisor 1.

Since $6X^2 + 2Y^2$ is even, then Z^2 is even, so Z is even.
Thus, $Z = 2U$ for some integer U, which gives $6X^2 + 2Y^2 = (2U)^2$ or $6X^2 + 2Y^2 = 4U^2$ and so $3X^2 + Y^2 = 2U^2$.
For $3X^2 + Y^2$ to be even, we must have X and Y both even or both odd.
If X and Y are both even, then X, Y, and Z, are all divisible by 2. This is impossible, since X, Y, and Z have no common divisor larger than 1.
Hence $X = 2A + 1$, $Y = 2B + 1$ for some integers A and B.
Therefore,

$$\begin{aligned} 3(2A+1)^2 + (2B+1)^2 &= 2U^2 \\ 12A^2 + 12A + 3 + 4B^2 + 4B + 1 &= 2U^2 \\ 6A^2 + 6A + 2B^2 + 2B + 2 &= U^2 \end{aligned}$$

Thus U is even, say $U = 2V$ for some integer V.
Then

$$\begin{aligned} 6A^2 + 6A + 2B^2 + 2B + 2 &= 4V^2 \\ 3A^2 + 3A + B^2 + B + 1 &= 2V^2 \\ 3A(A+1) + B(B+1) + 1 &= 2V^2 \end{aligned}$$

Since A and $A + 1$ are consecutive integers, then one of them is even.
Thus $A(A + 1)$ is even.
Similarly, $B(B + 1)$ is even.
This means that $3A(A + 1) + B(B + 1) + 1$ is odd while $2V^2$ is even.
This is a contradiction, so no solution other than $(x, y, z) = (0, 0, 0)$ exists. □

5. *1982 Pascal Contest, Question 8*
A pole is painted in white, green, and blue sections. If one-third of the pole is white and one-quarter of the pole is green, then the fraction of the pole that is blue is

(A) $\frac{6}{7}$ **(B)** $\frac{11}{12}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{5}{7}$

Solution
The pole is made up of three sections.
Let the fractions of the pole that are white, green, and blue be W, G, B, respectively.
We can say $W + G + B = 1$.
Substituting the given values for the white and green sections we find that

$$\begin{aligned} \frac{1}{3} + \frac{1}{4} + B &= 1 \\ \frac{4}{12} + \frac{3}{12} + B &= \frac{12}{12} \\ B &= \frac{12}{12} - \frac{7}{12} \\ B &= \frac{5}{12} \end{aligned}$$

ANSWER: (D)